Finite Automata

Part Three

New Stuff!

NFAs and DFAs

- We know that any language for which there exists a DFA can also be recognized by an NFA.
- Why?
  - Every DFA essentially already *is* an NFA!

NFAs and DFAs

- We know that any language for which there exists a DFA can also be recognized by an NFA.
- Why?
  - Every DFA essentially already *is* an NFA!
- ***Question**:* Can any language recognized by an NFA also be recognized by a DFA?

NFAs and DFAs

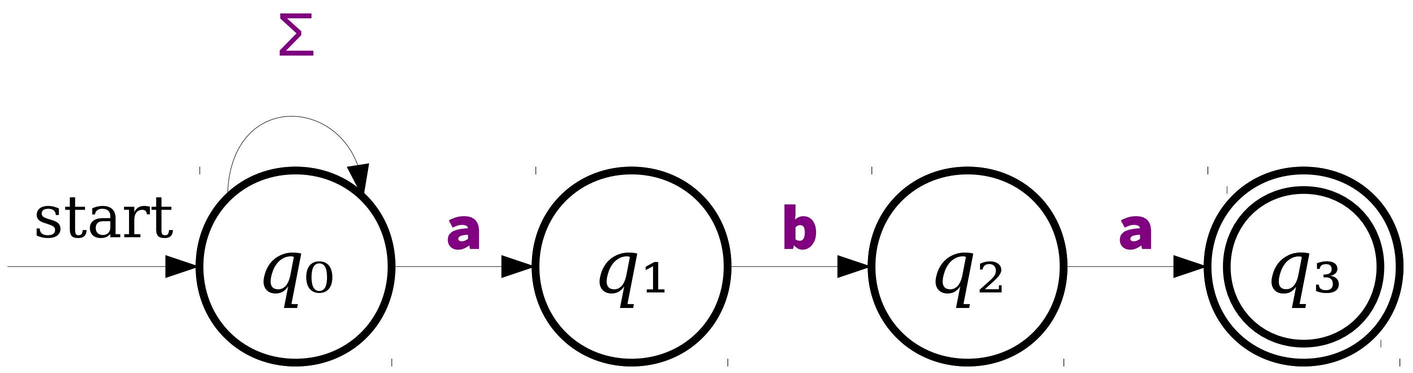
- We know that any language for which there exists a DFA can also be recognized by an NFA.
- Why?
  - Every DFA essentially already *is* an NFA!
- ***Question:*** Can any language recognized by an NFA also be recognized by a DFA?
- Surprisingly, the answer is ***yes***!

NFAs and DFAs

- ***Question:*** Can any language recognized by an NFA also be recognized by a DFA?
- Surprisingly, the answer is ***yes***!
  - To prove this, we need to:
    - Pick an arbitrary language for which an NFA exists
    - Describe how we would construct a DFA with the same language (in a generalizable way)
    - For the next few slides, we'll ponder how to approach that…

Thought Experiment:

How would you simulate an NFA in software?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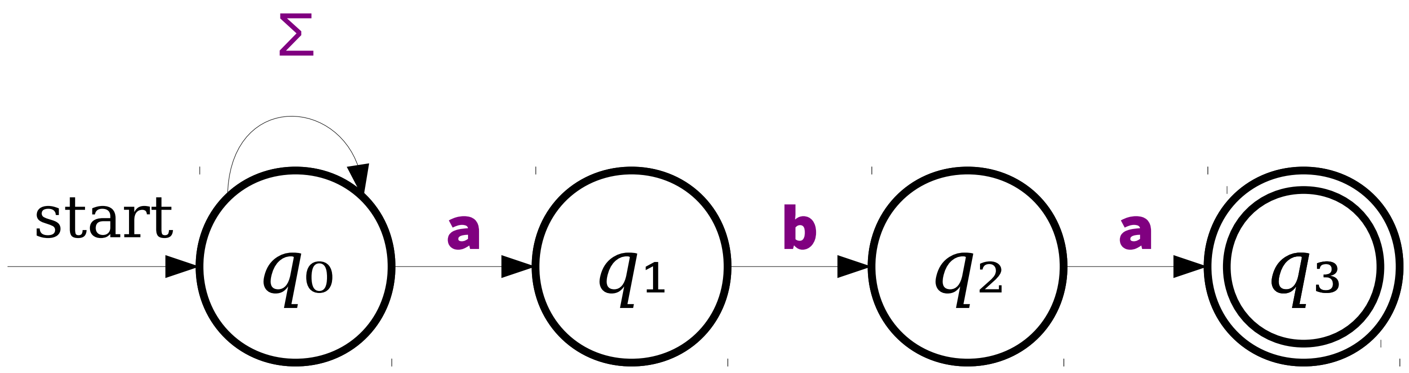

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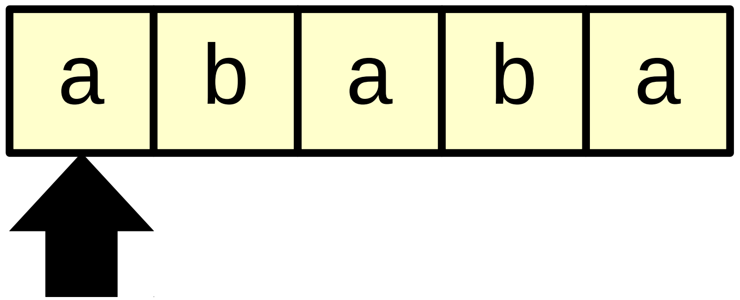

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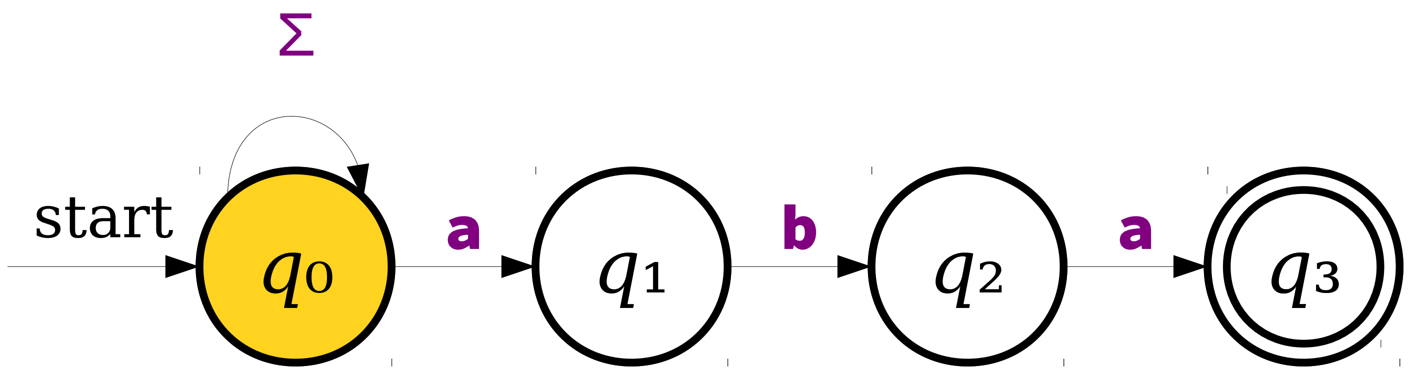
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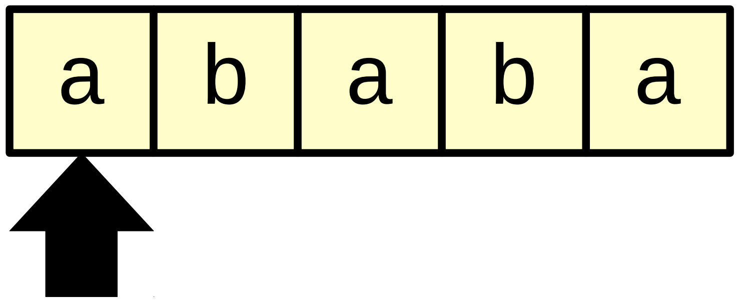
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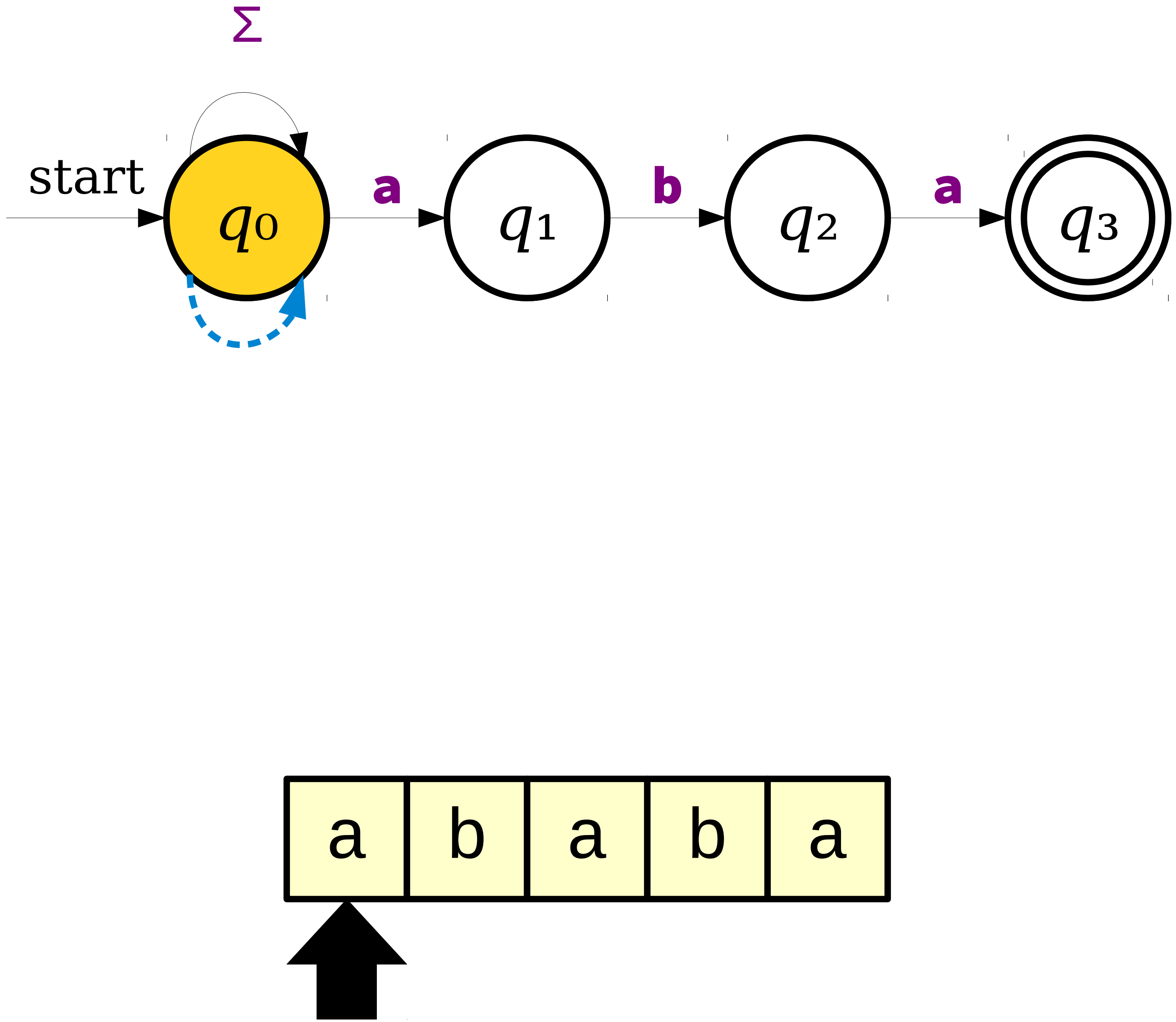
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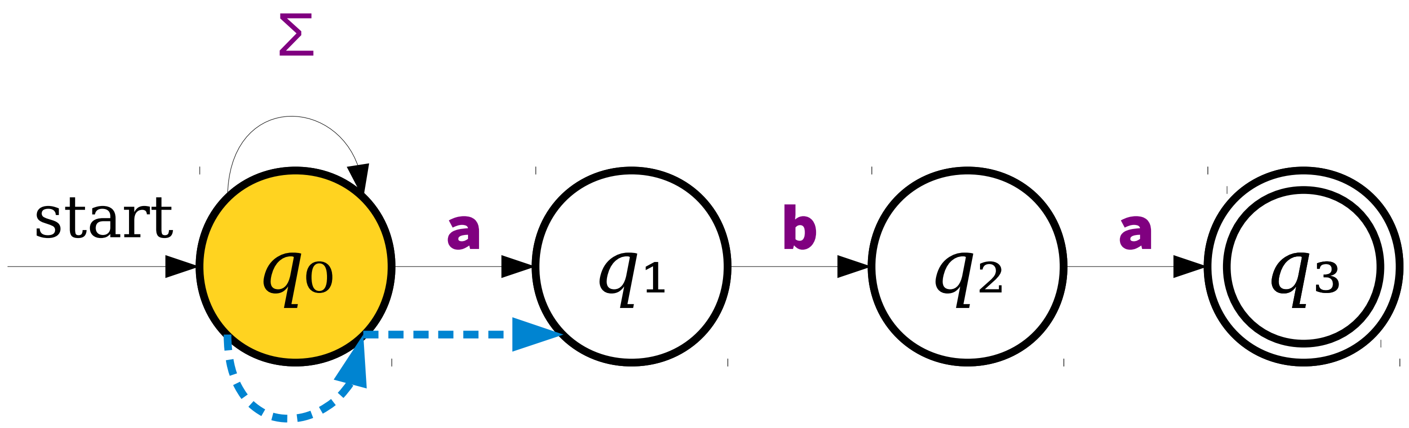
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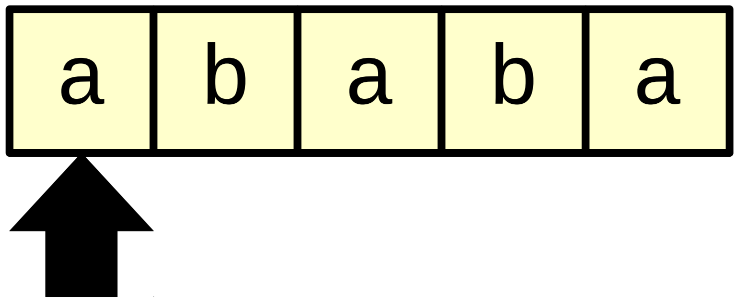
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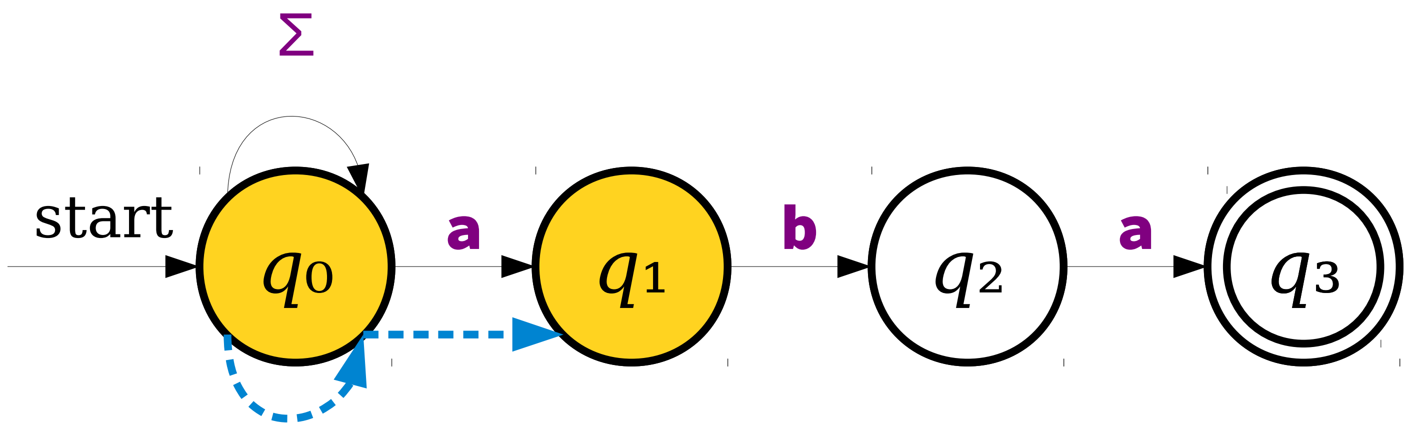
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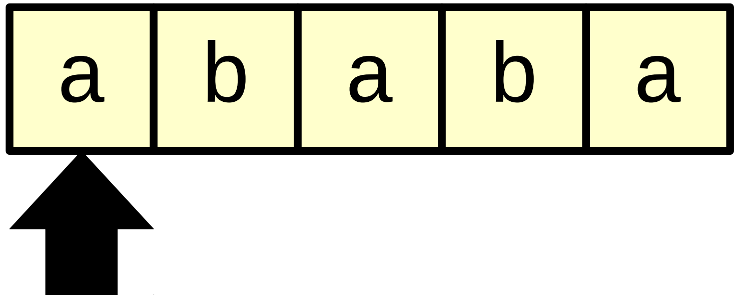
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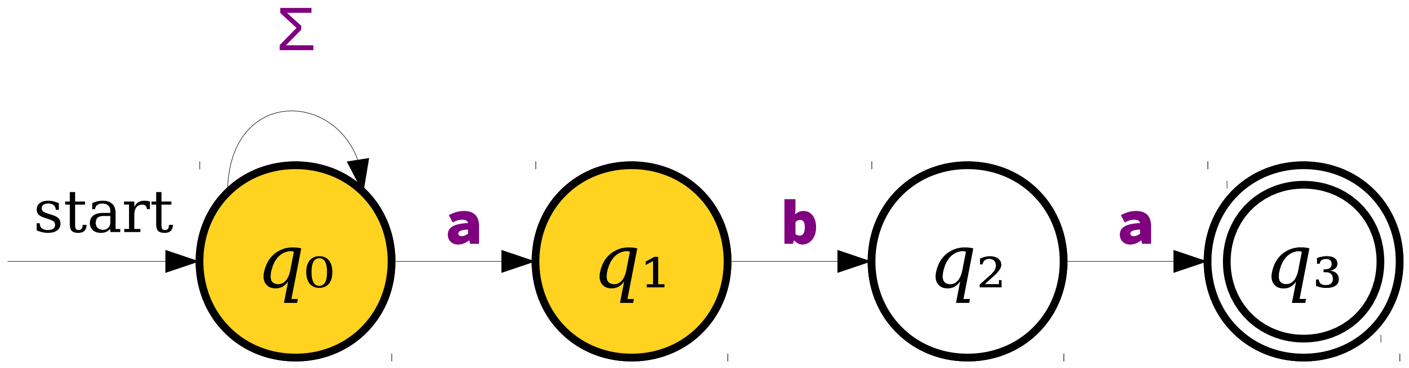
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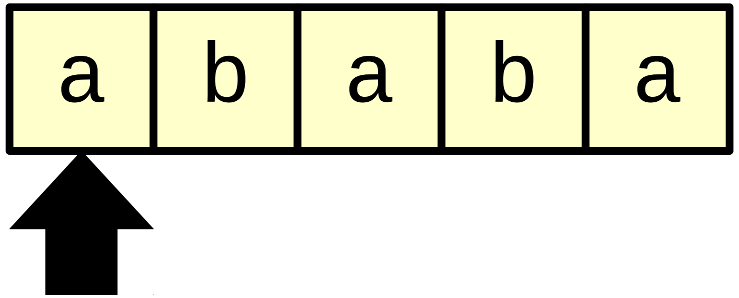
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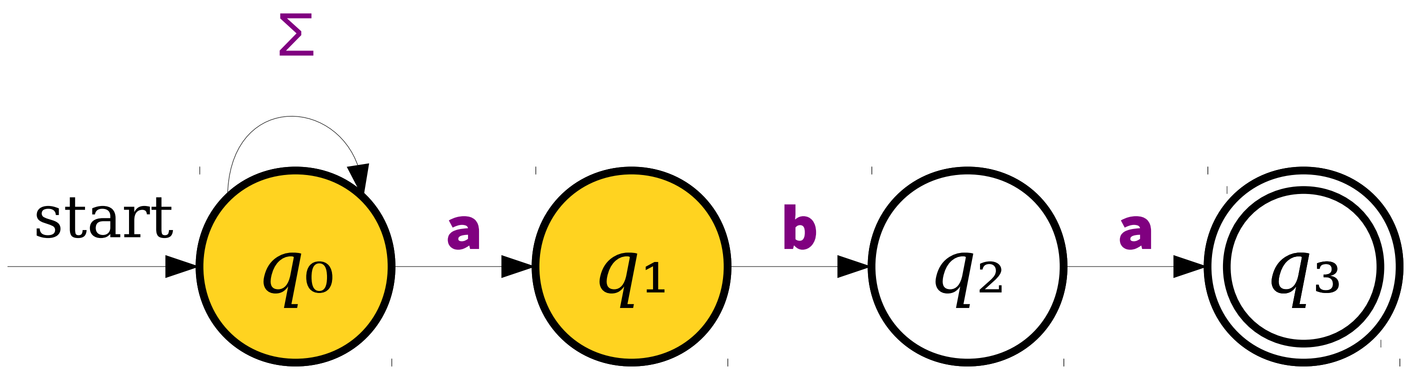
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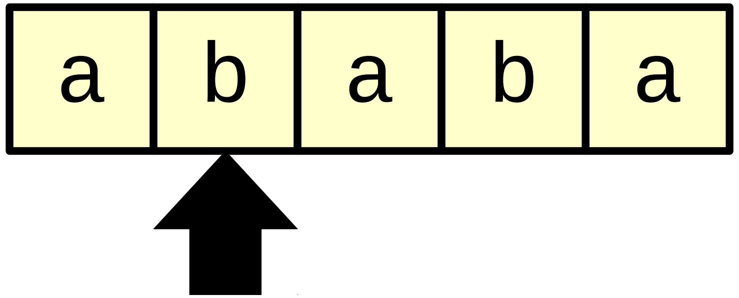
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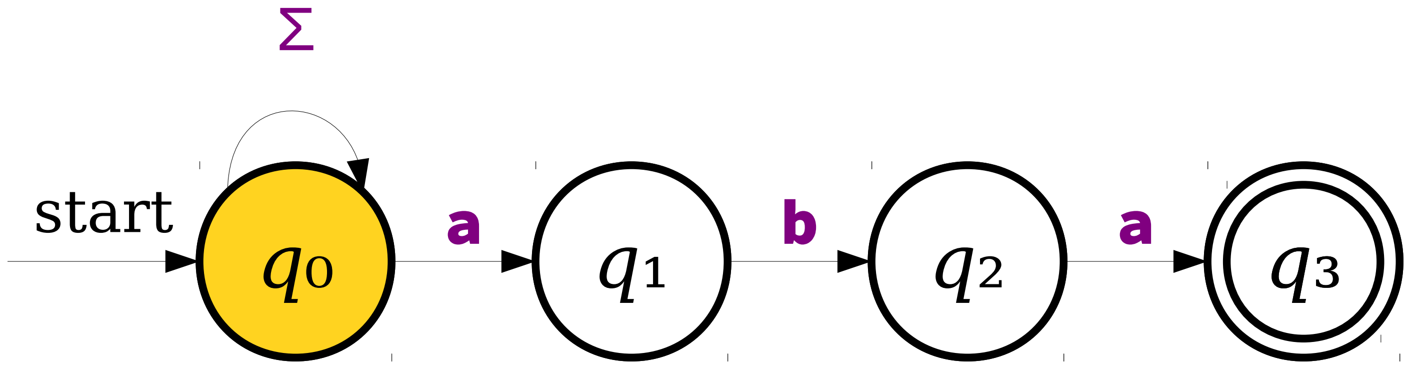
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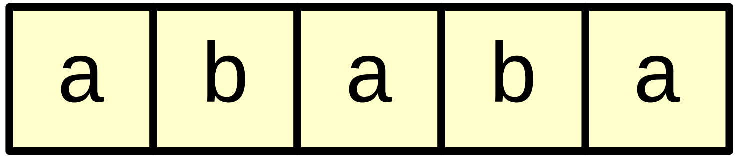
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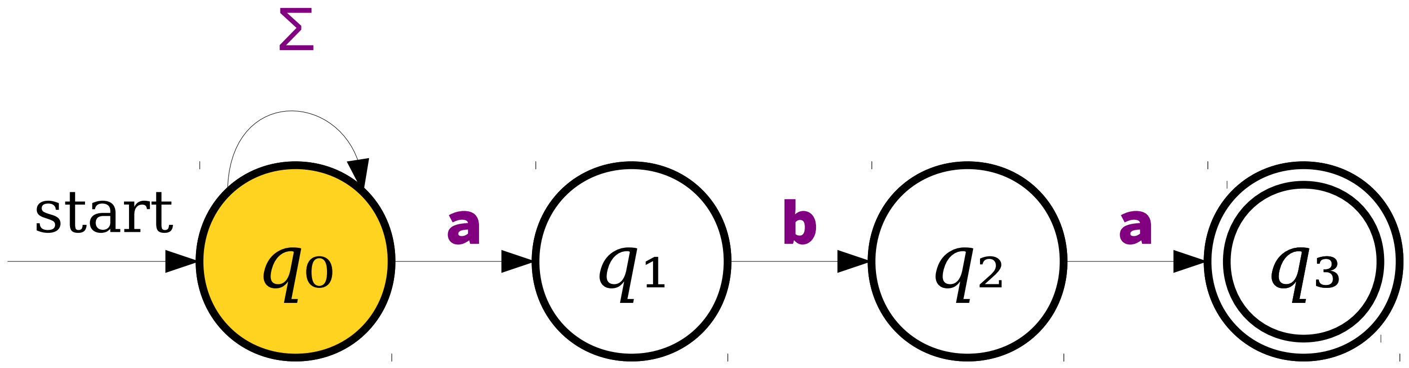
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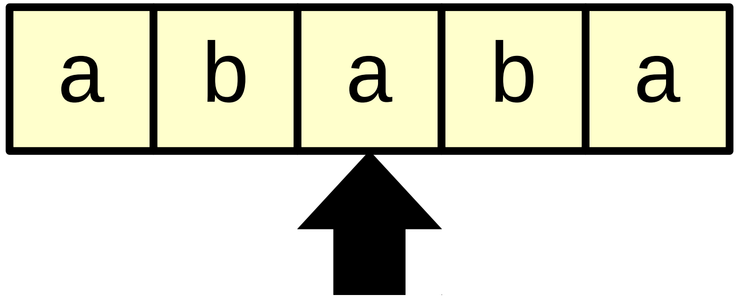
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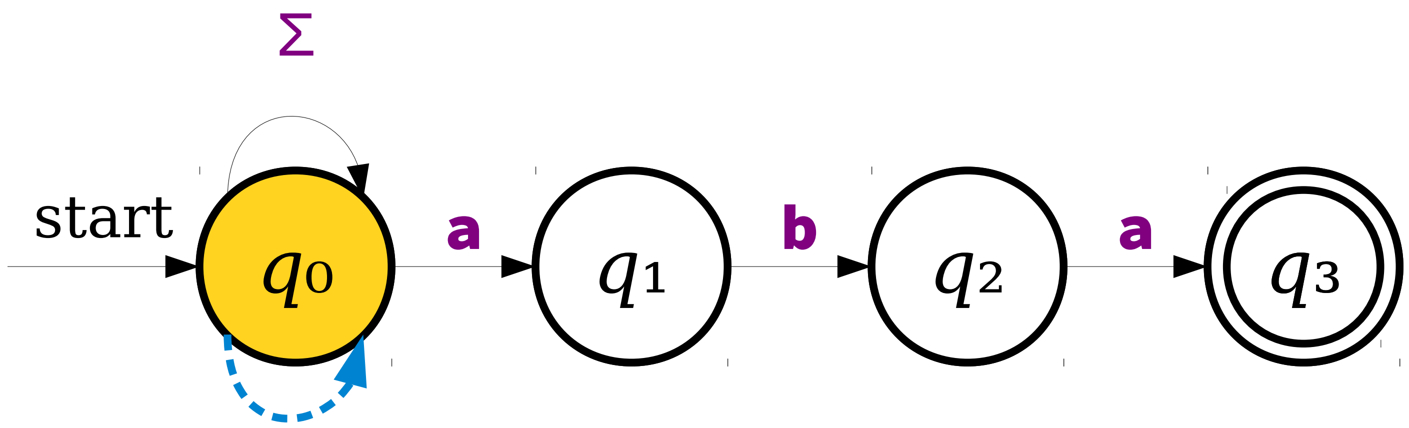

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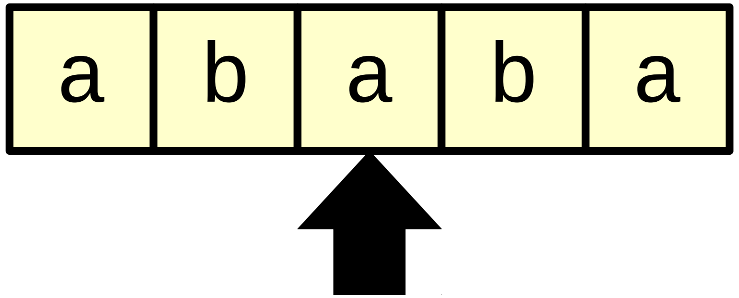

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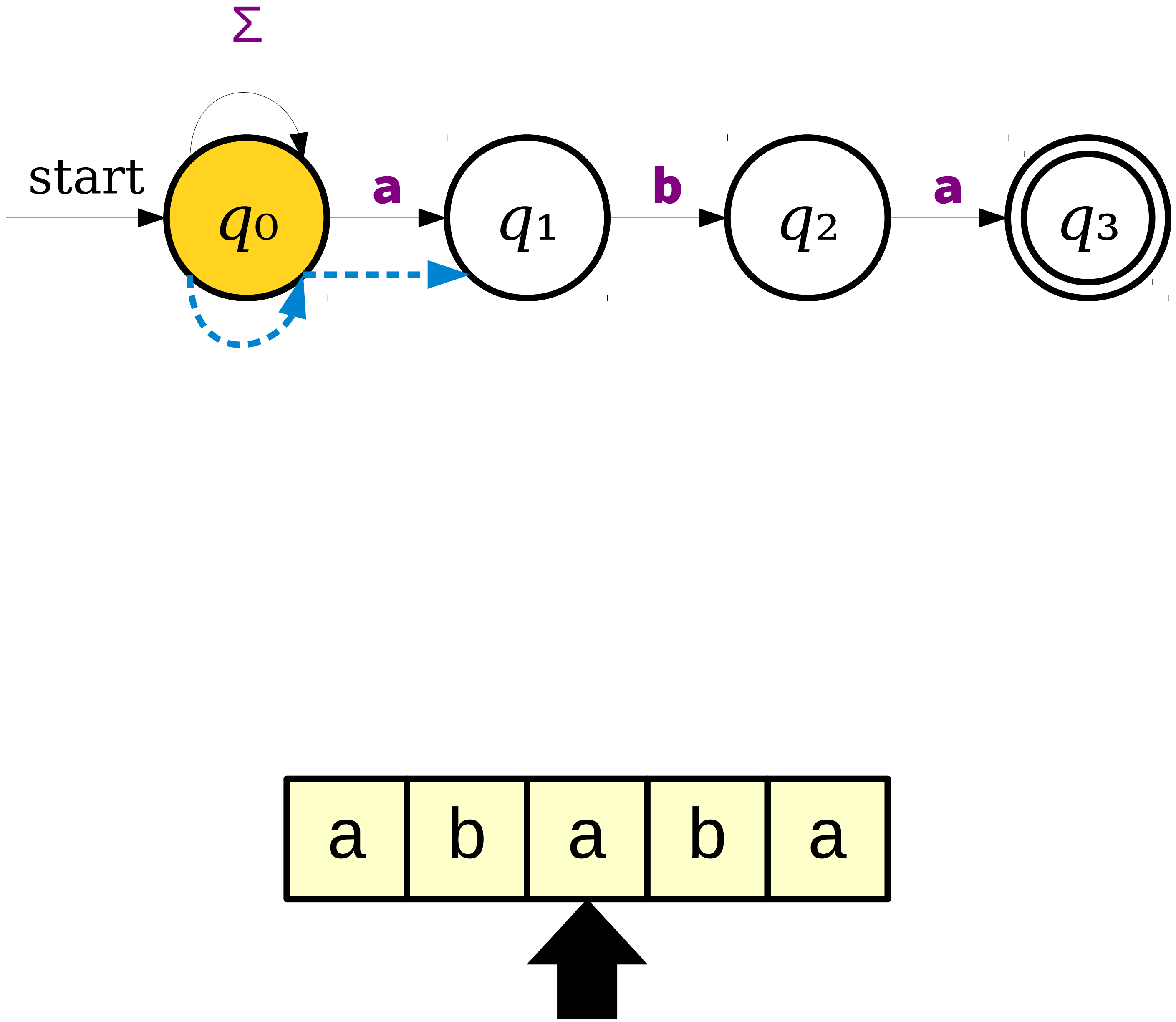
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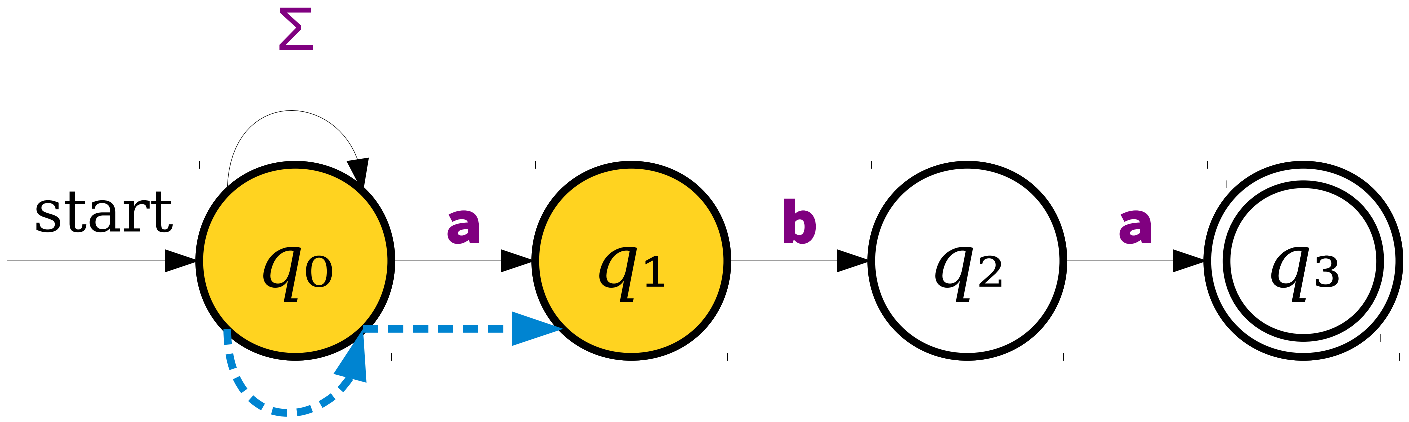
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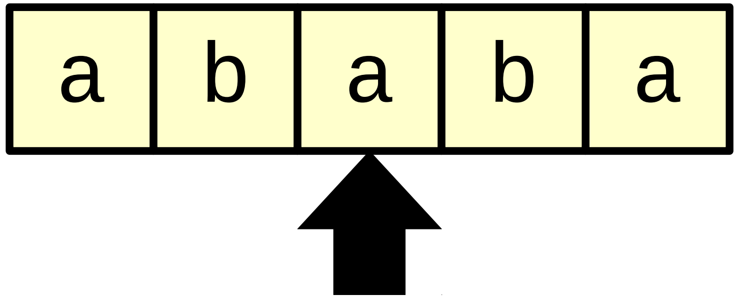
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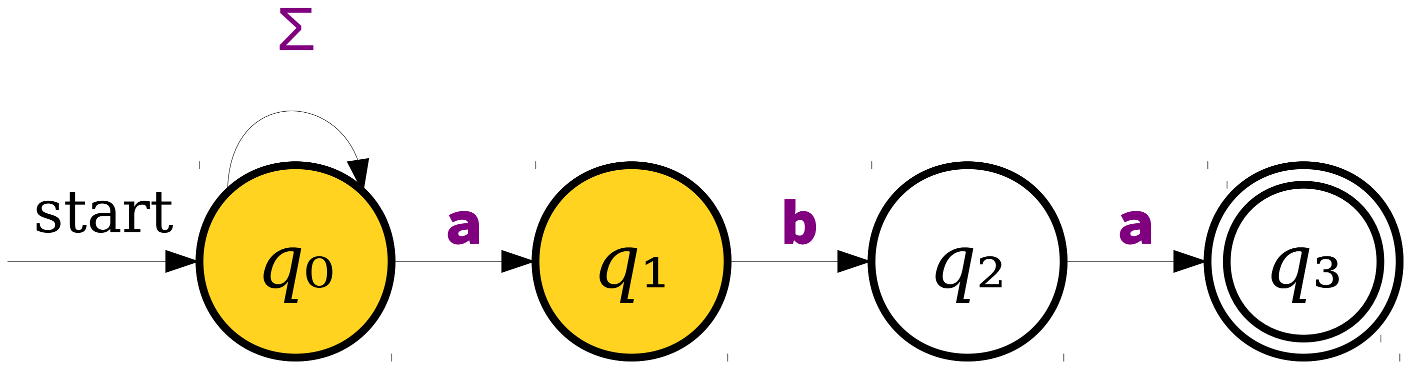
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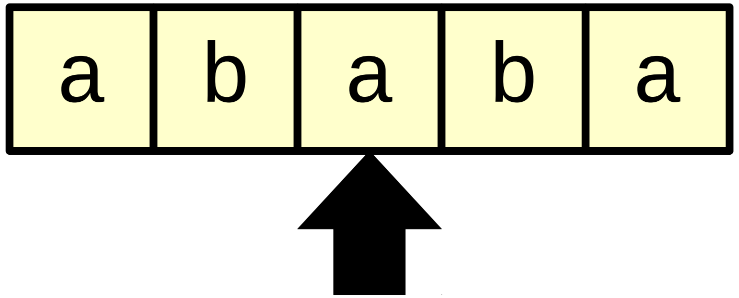
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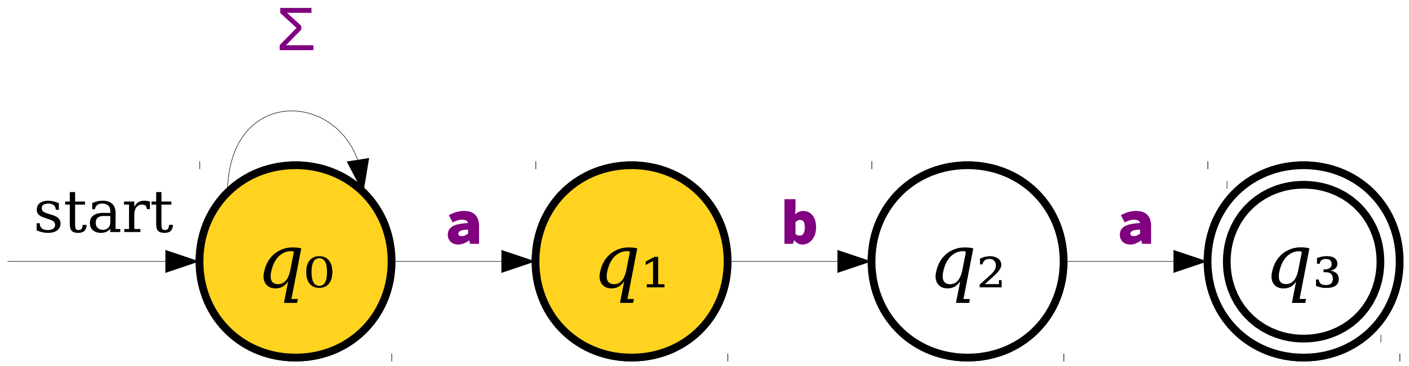

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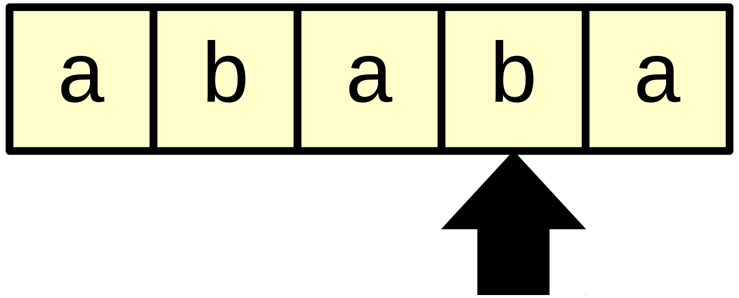

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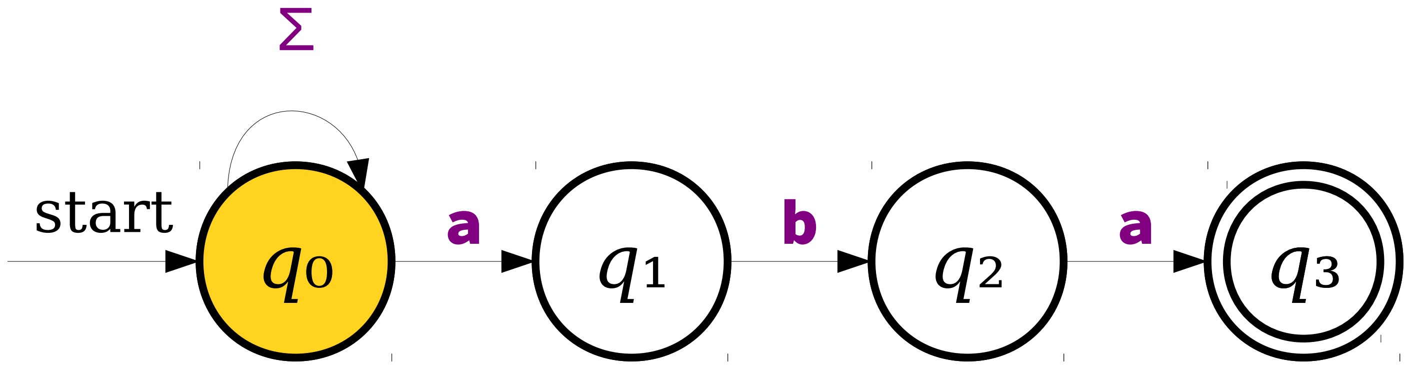
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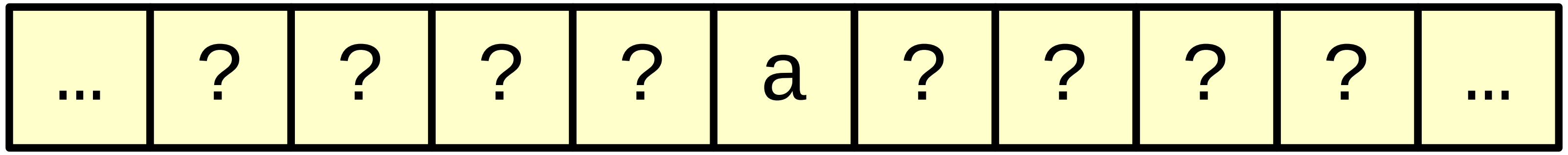
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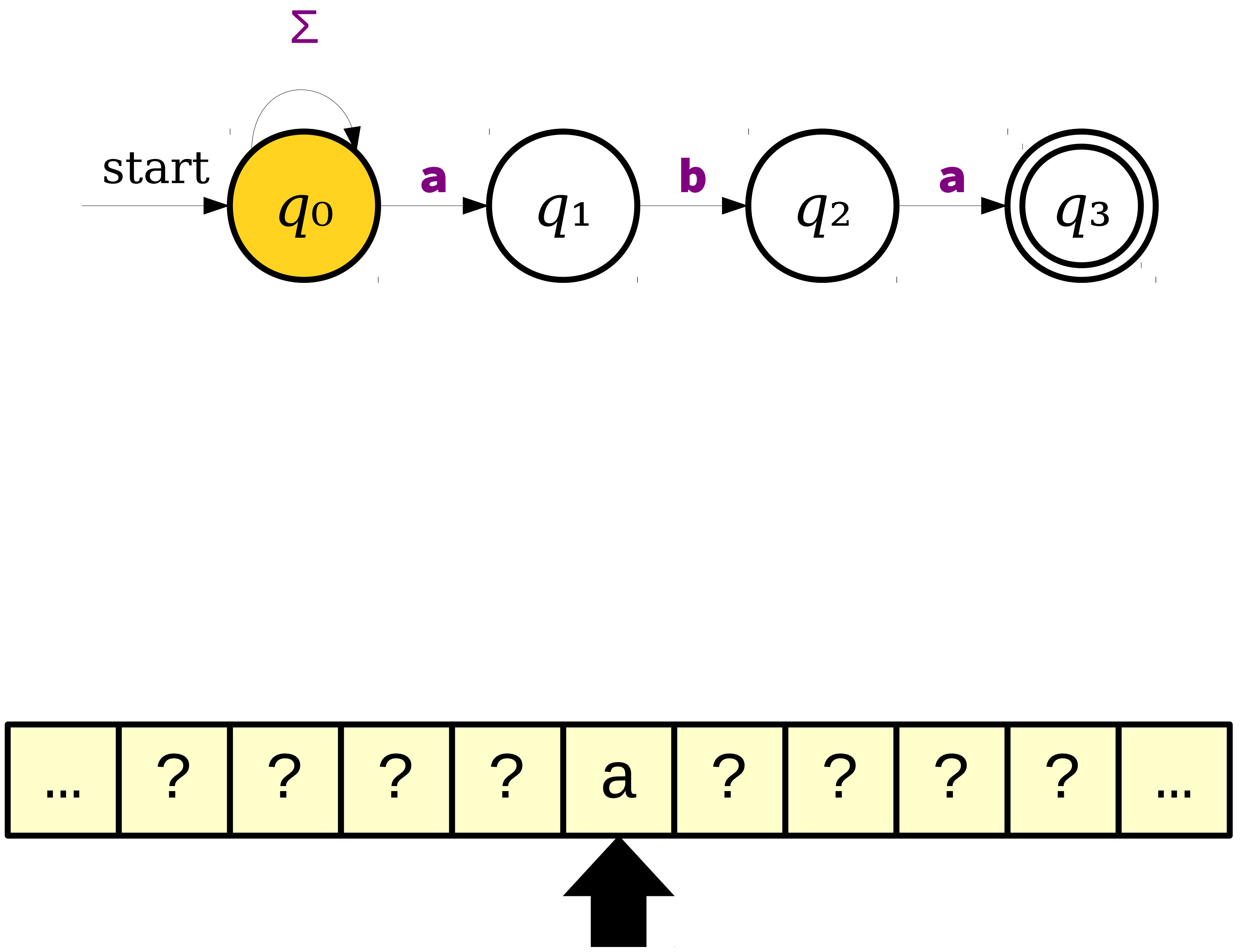

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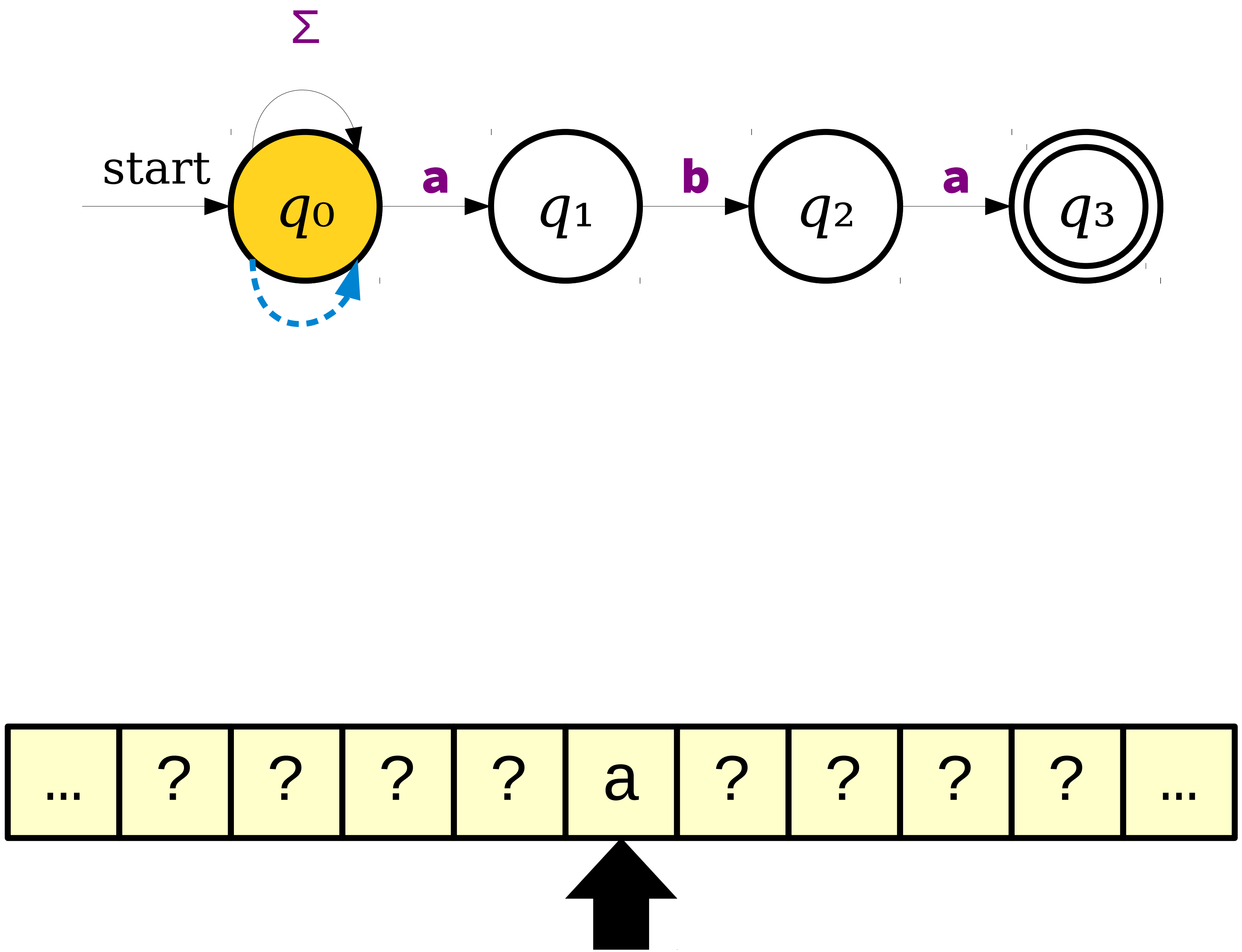
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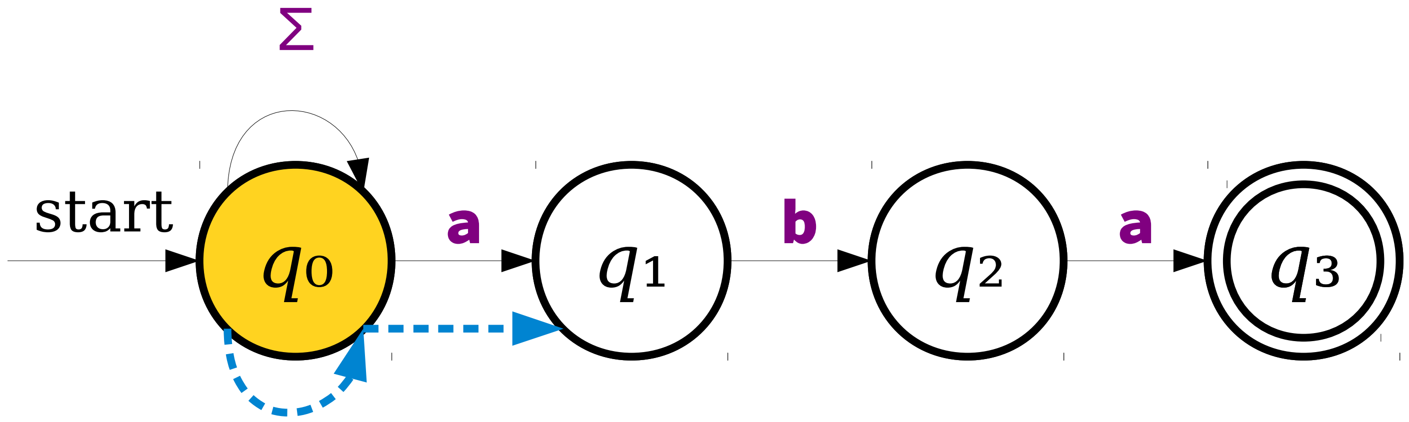
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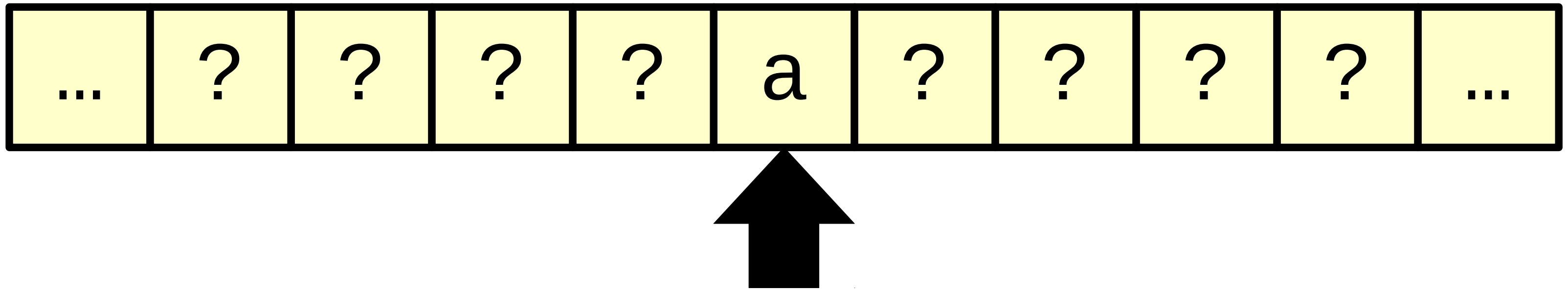
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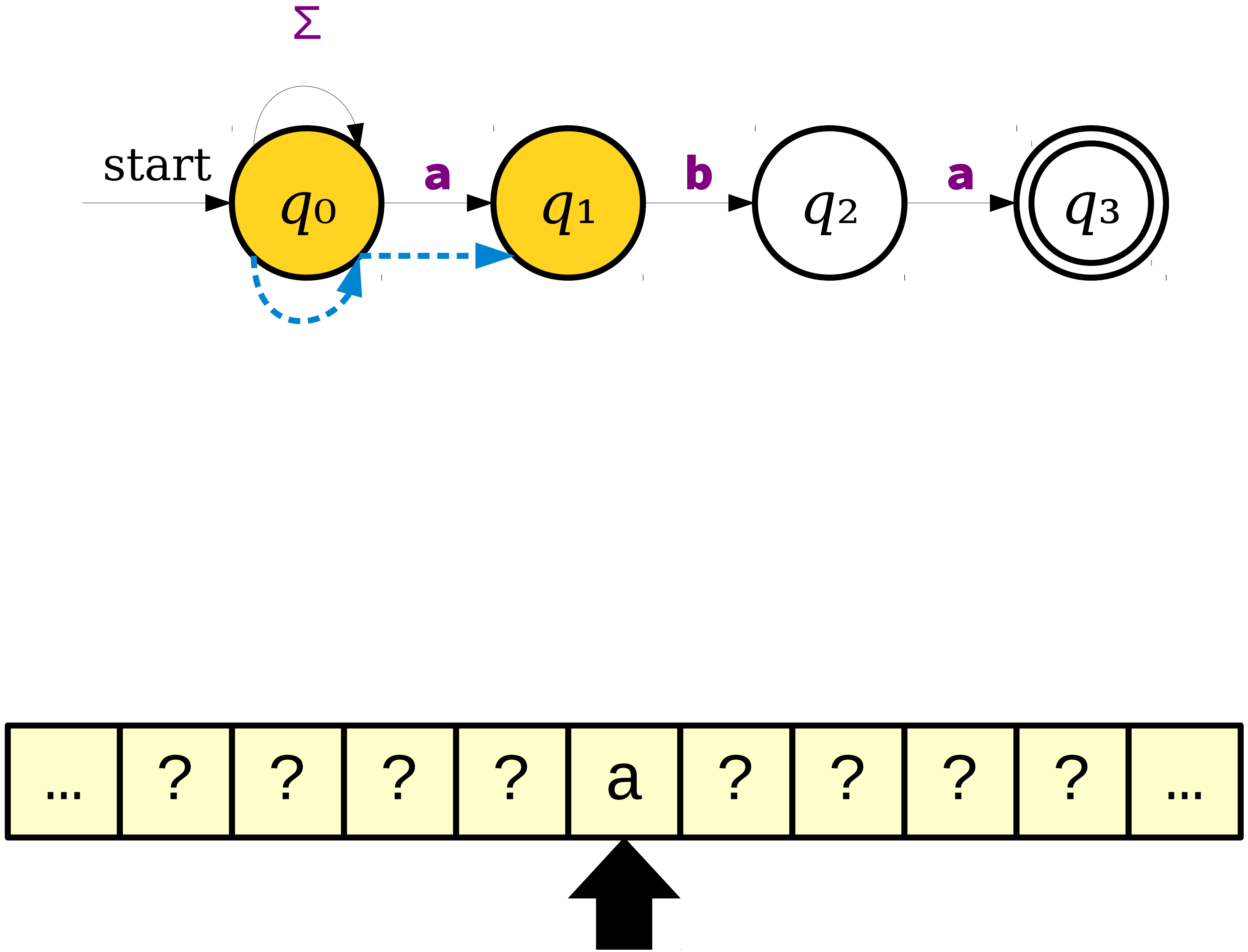

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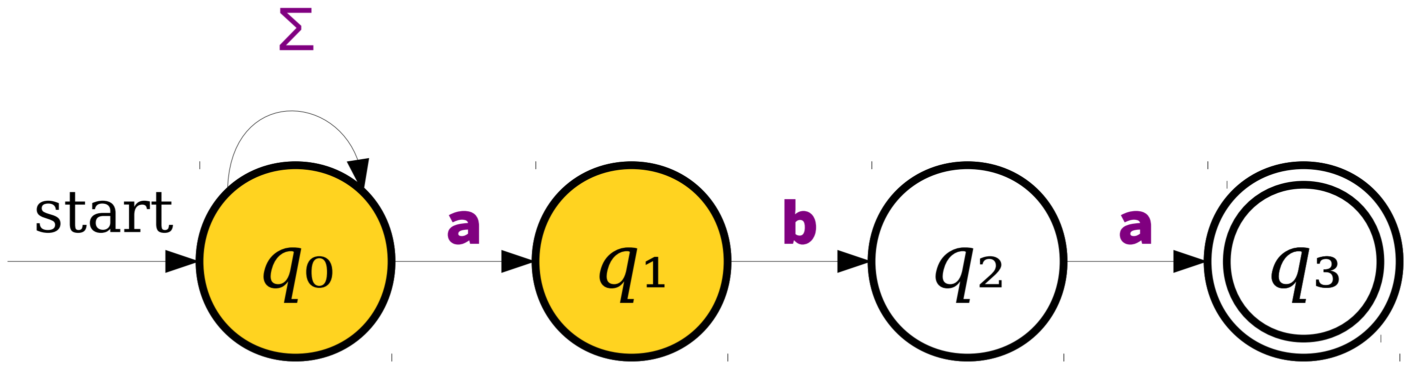
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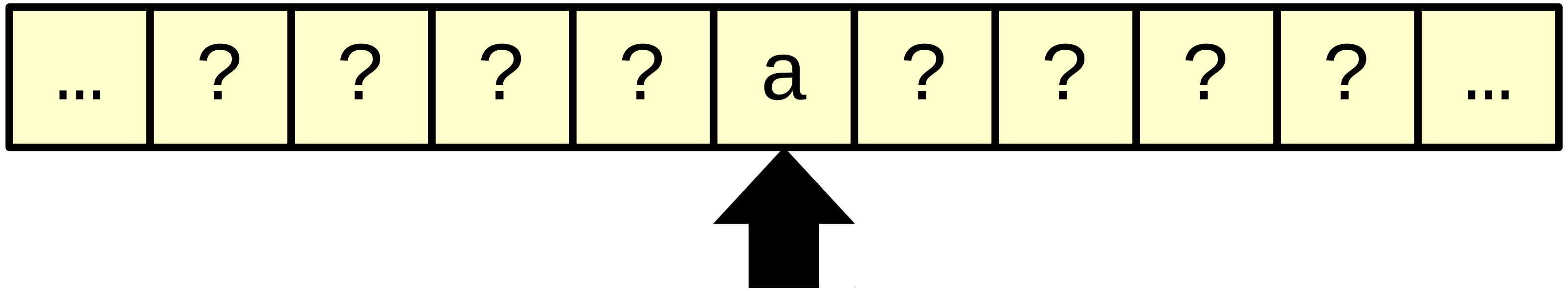
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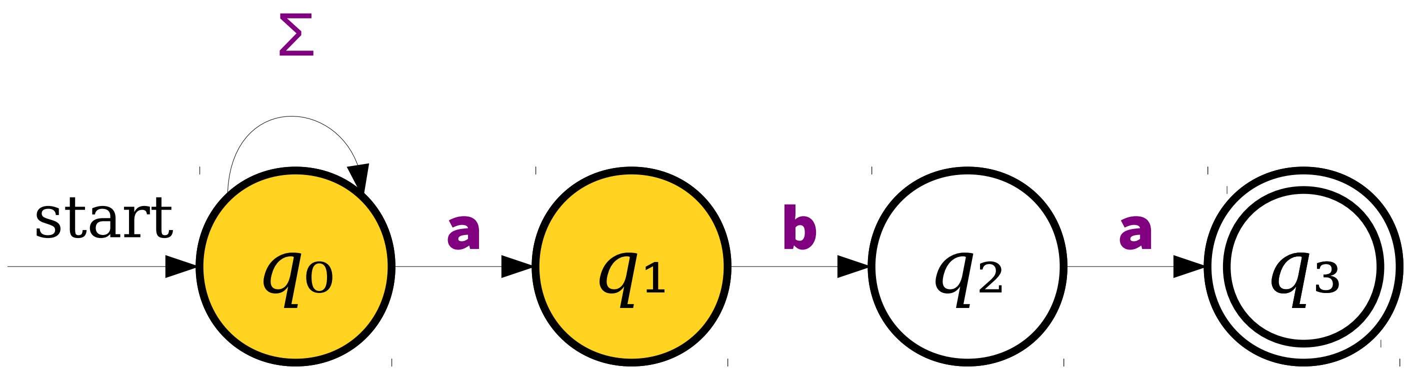
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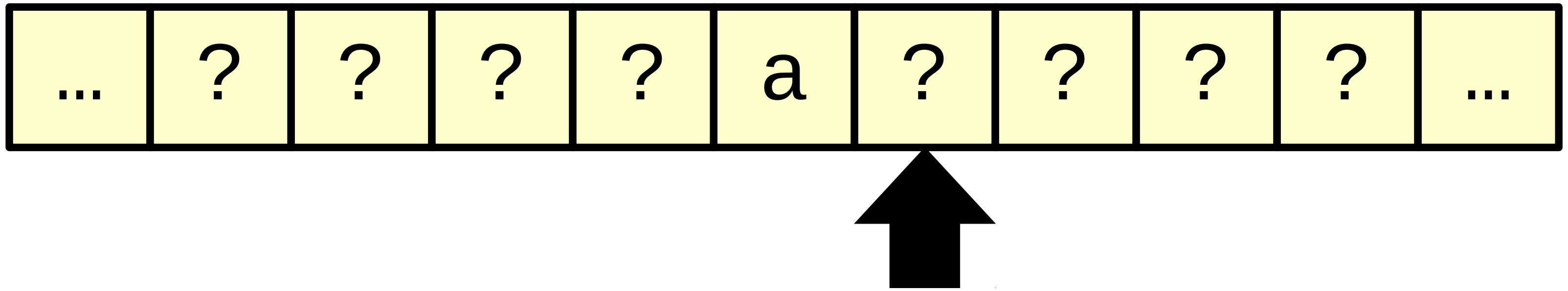
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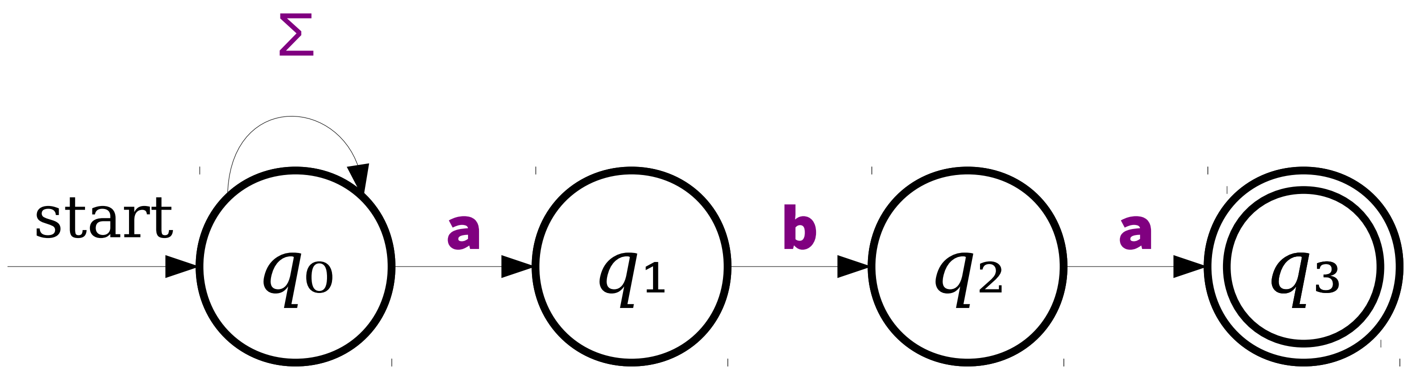

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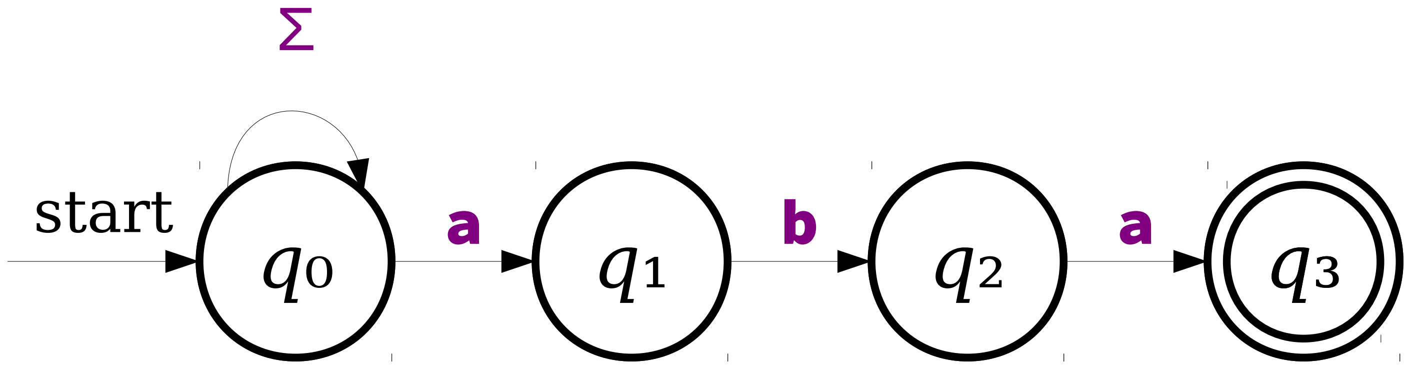

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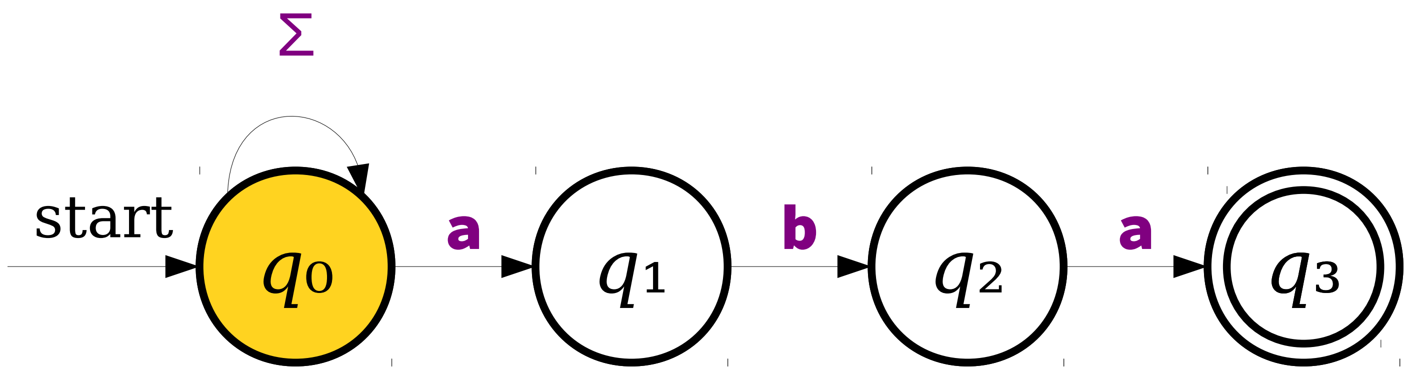

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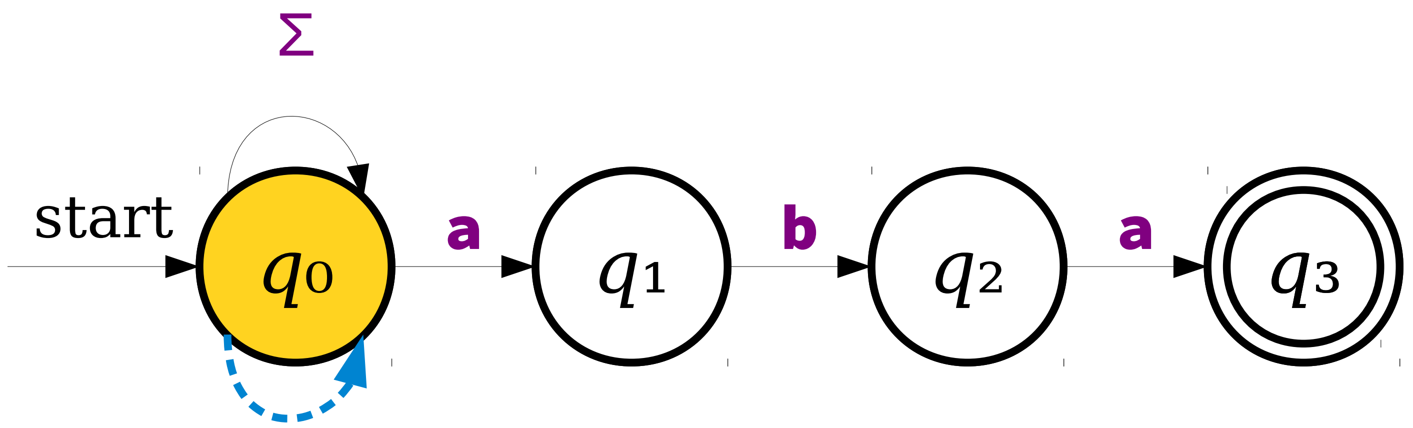

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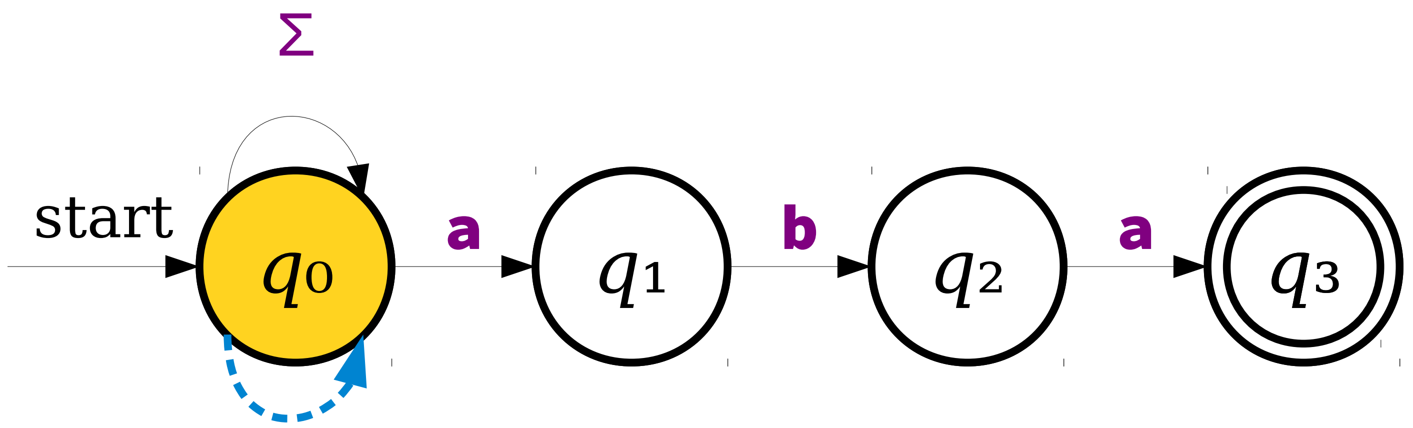

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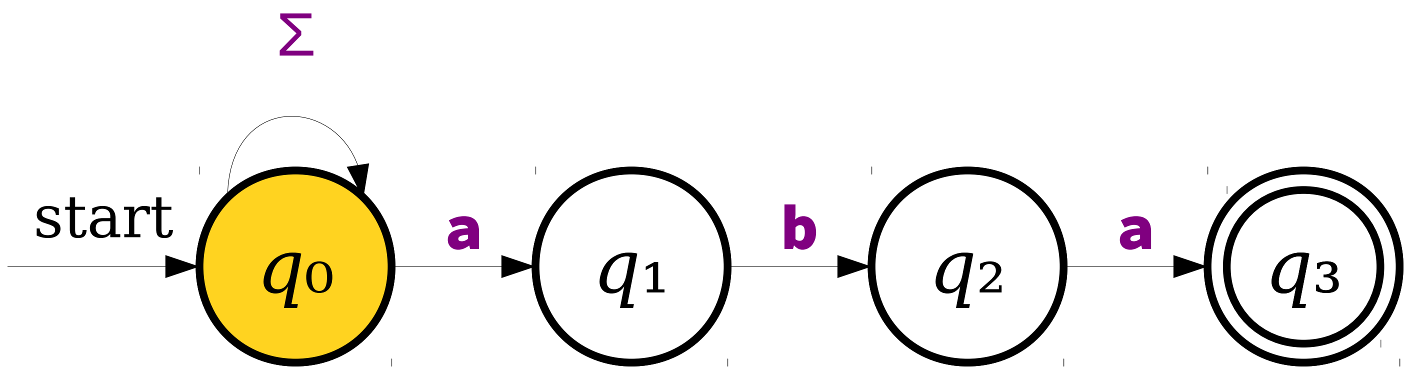

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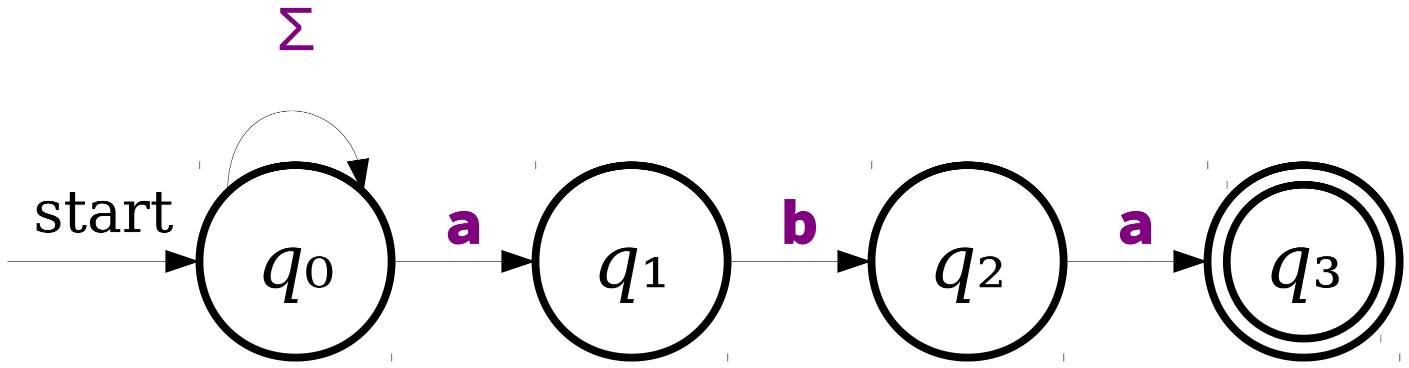

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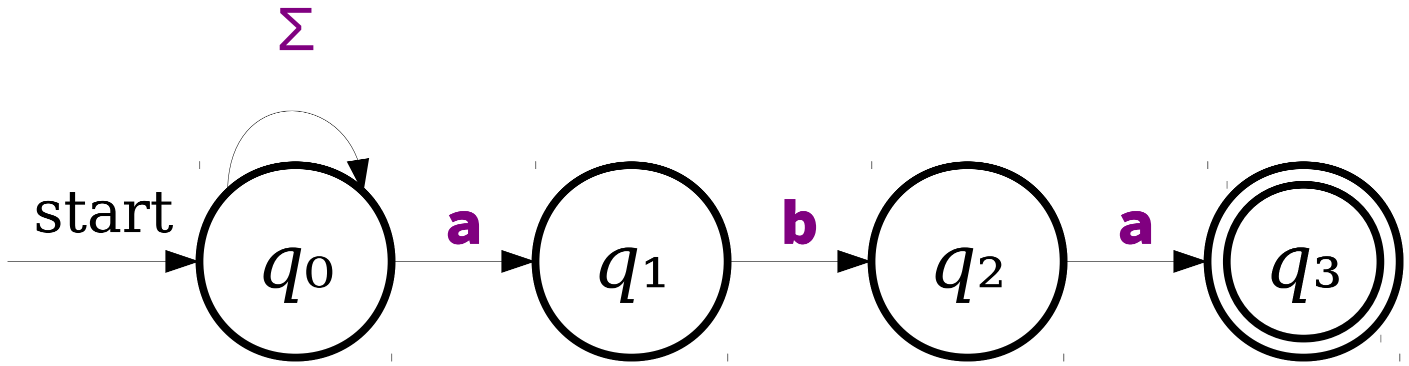

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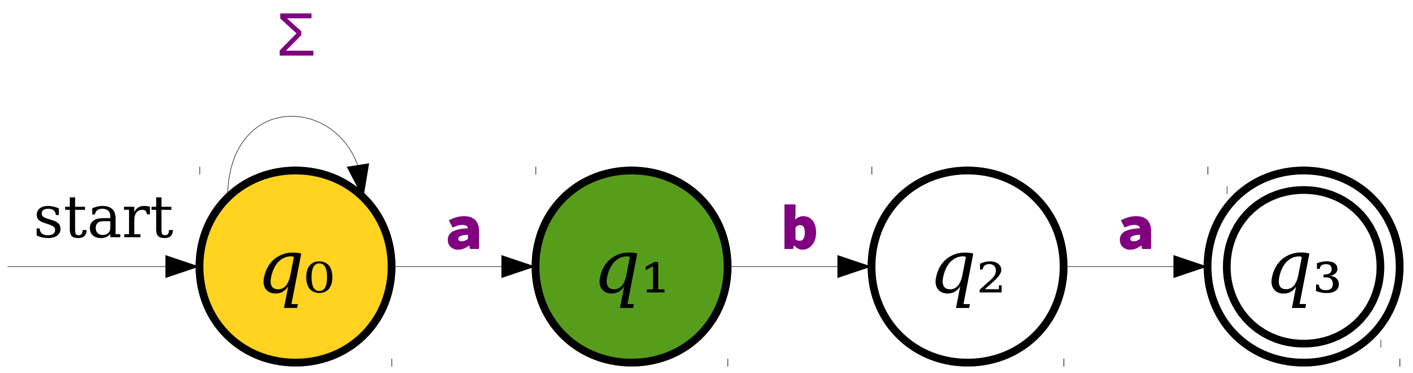

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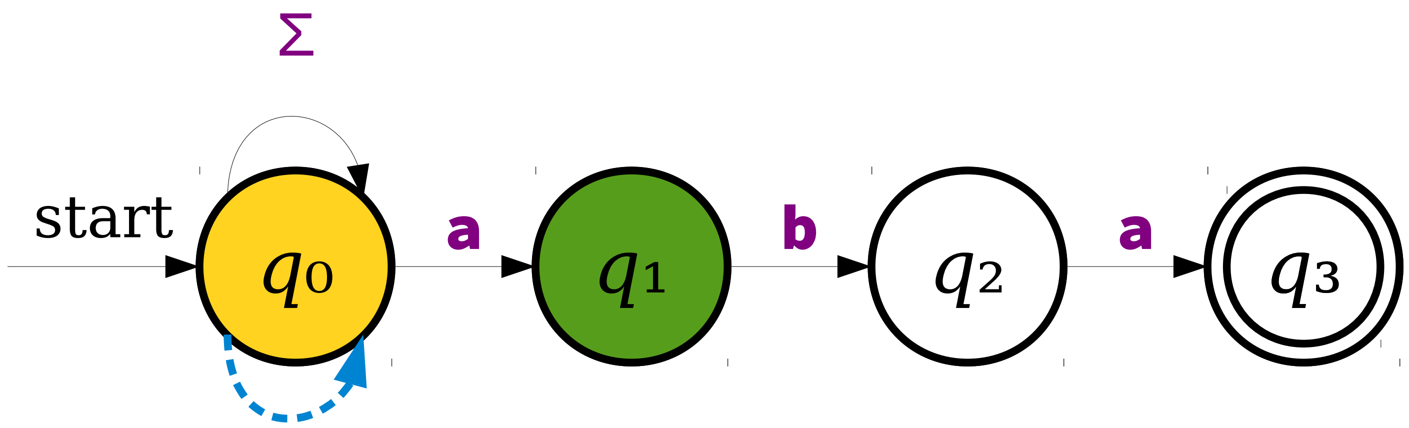

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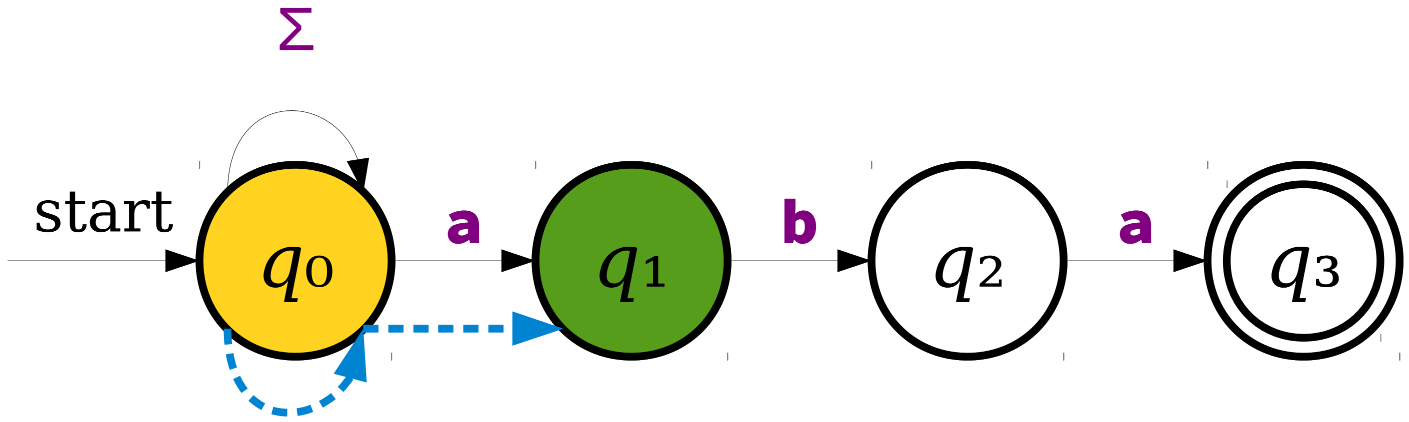

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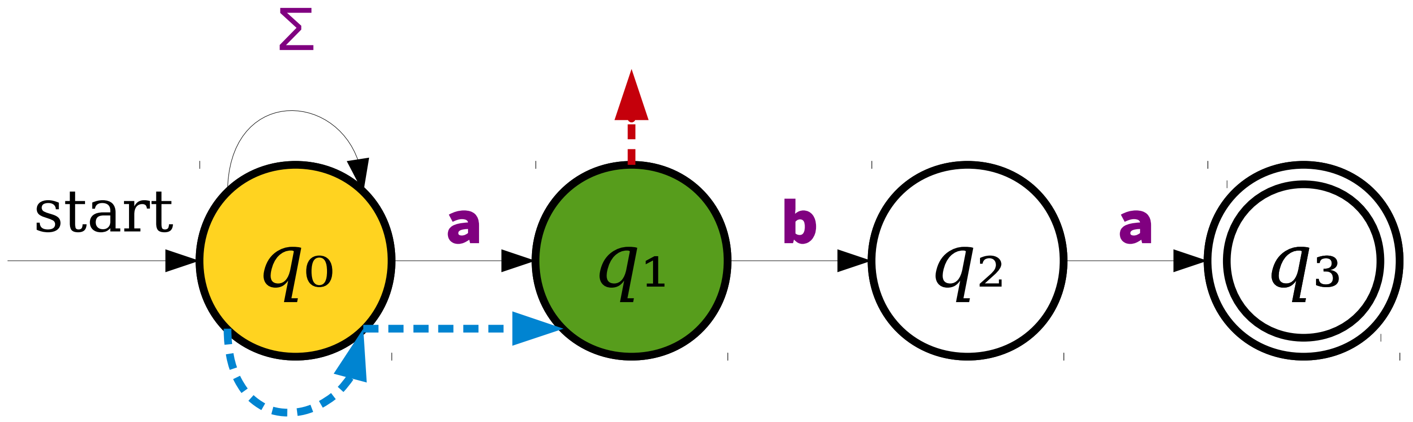

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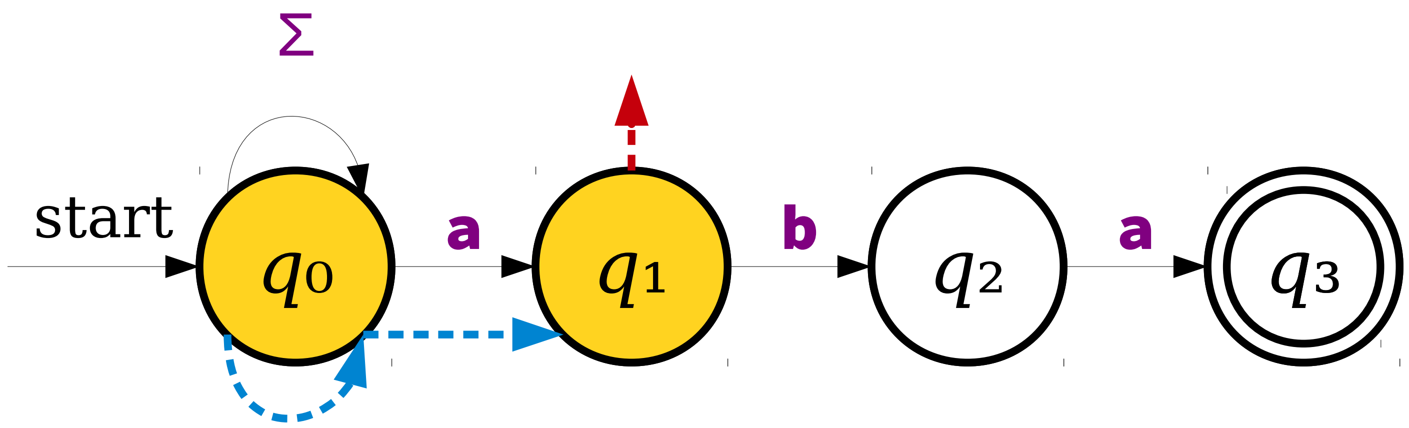

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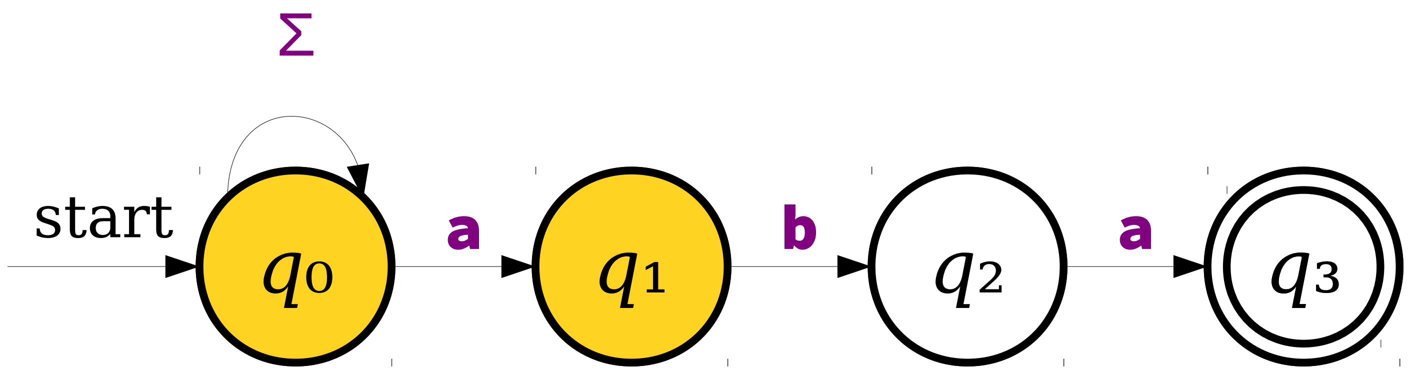

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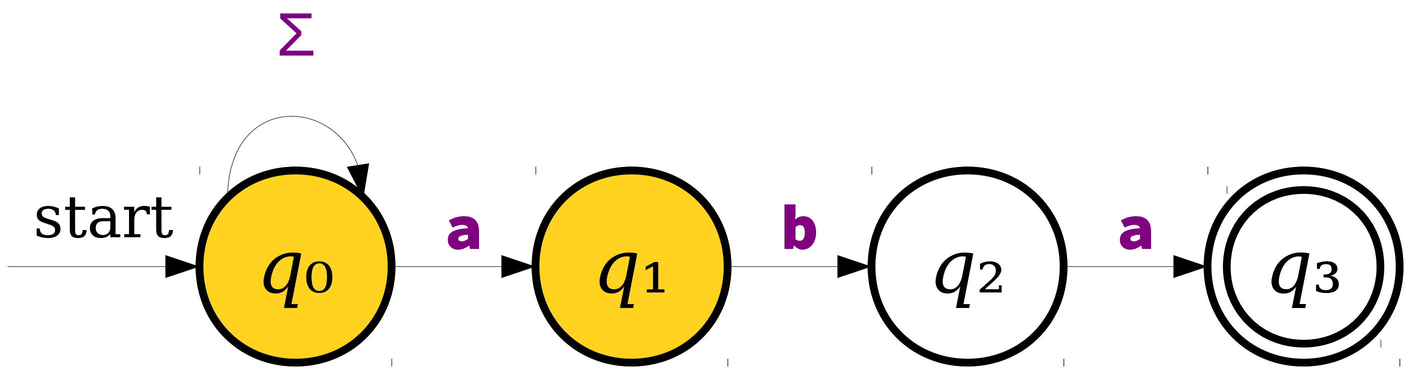

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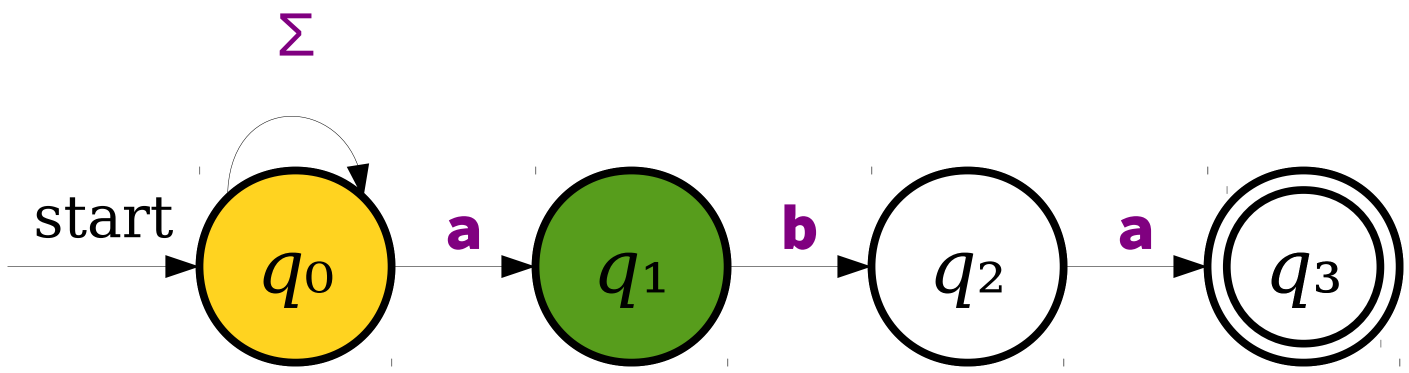

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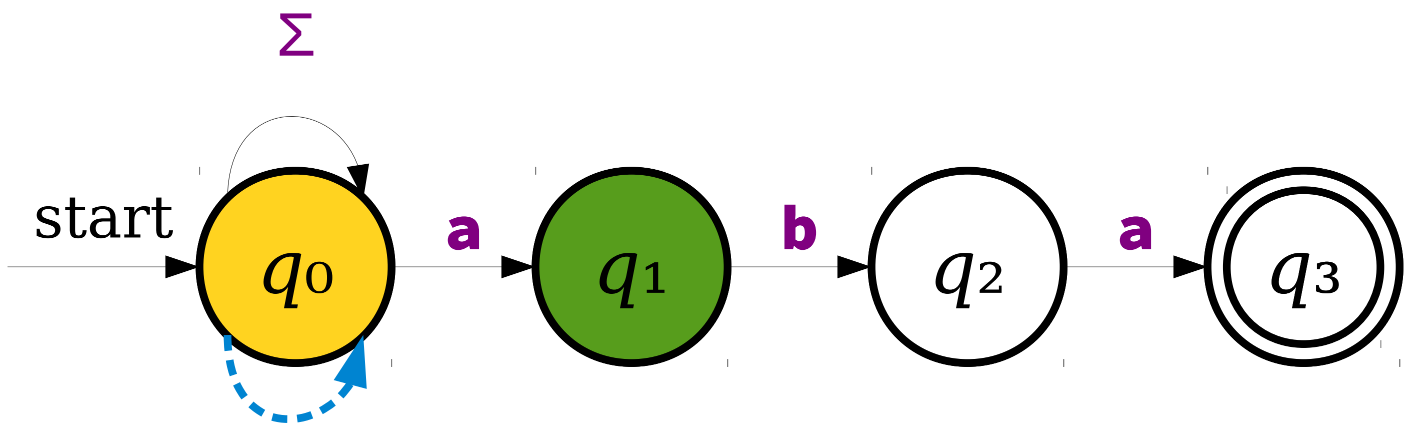

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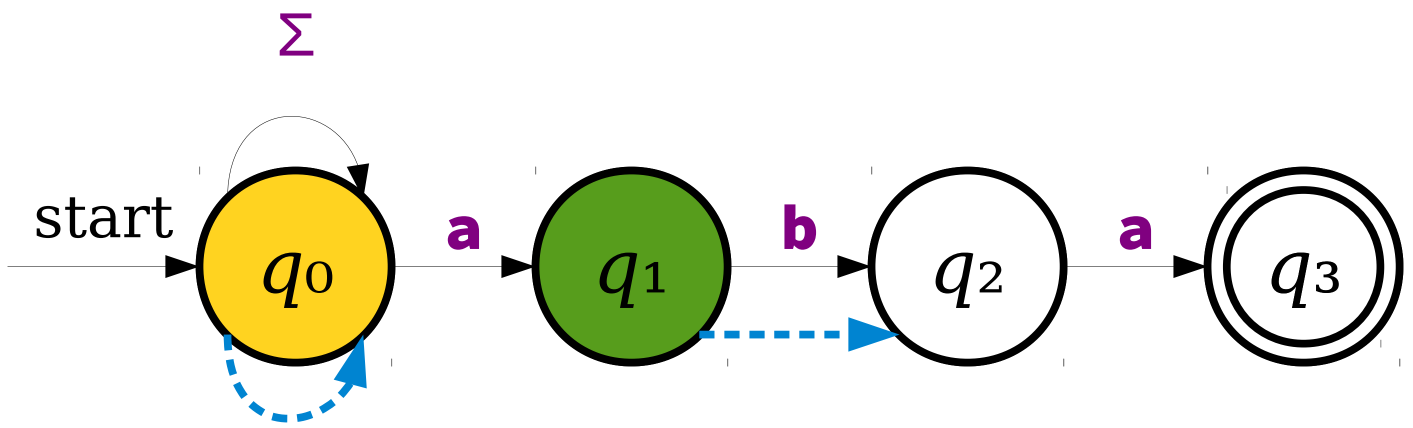

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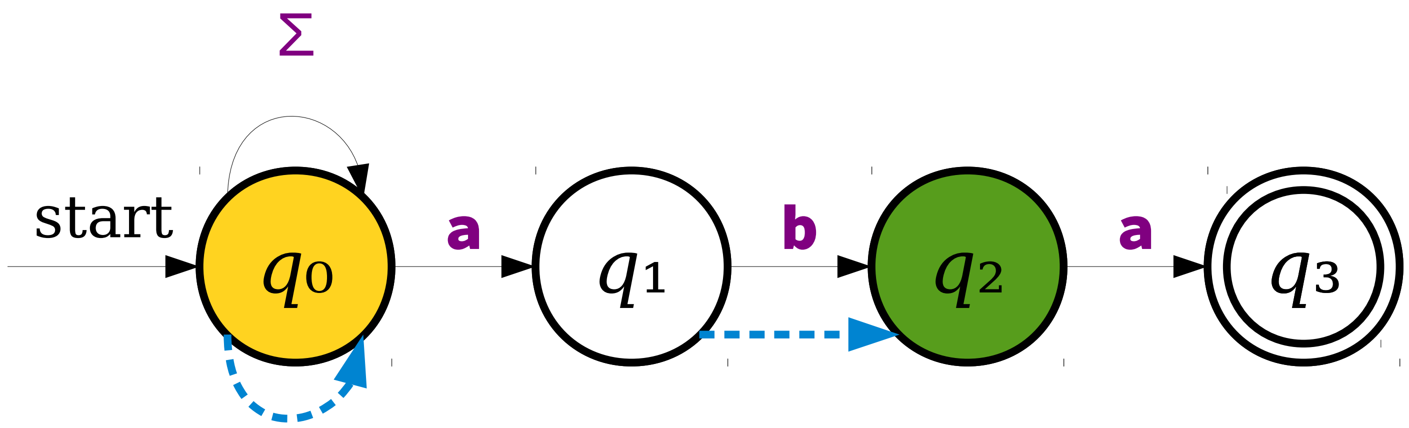

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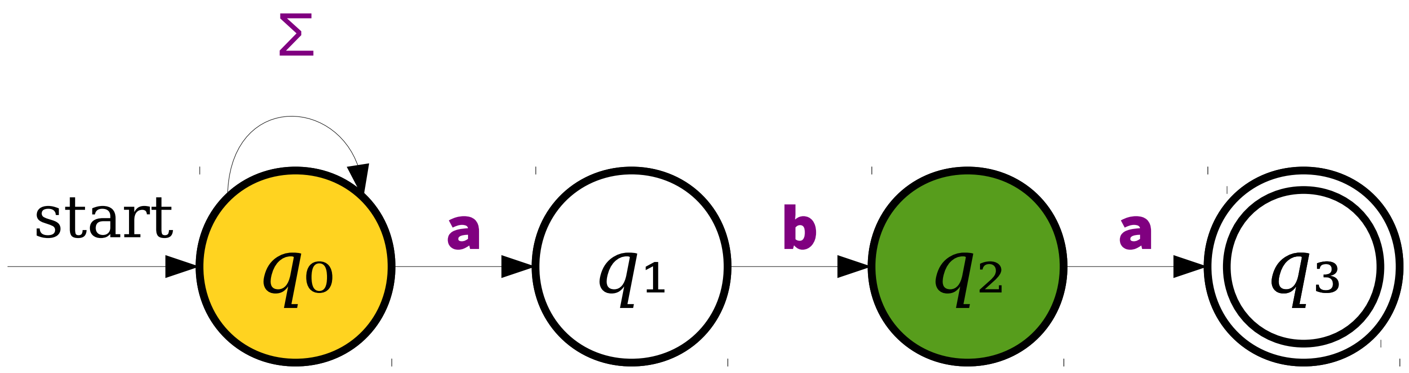

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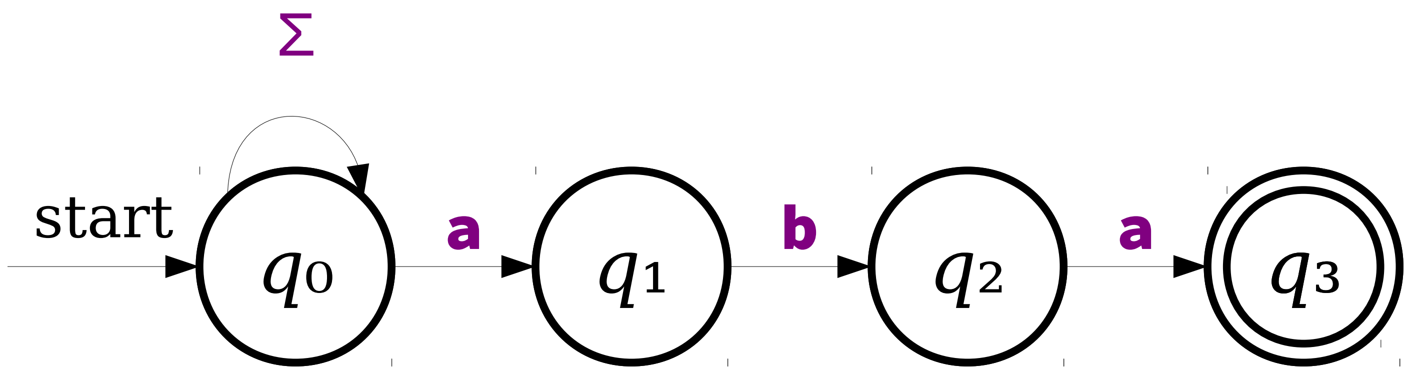

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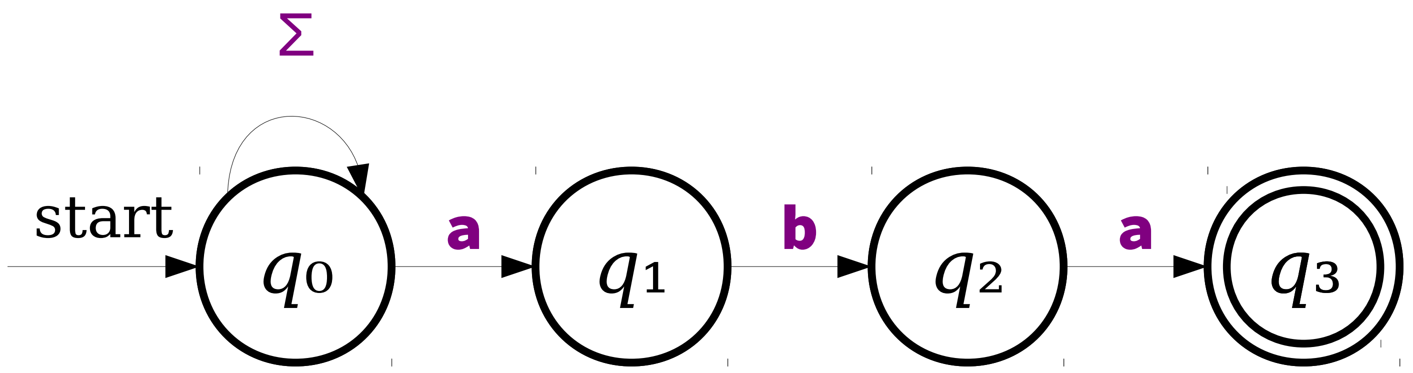

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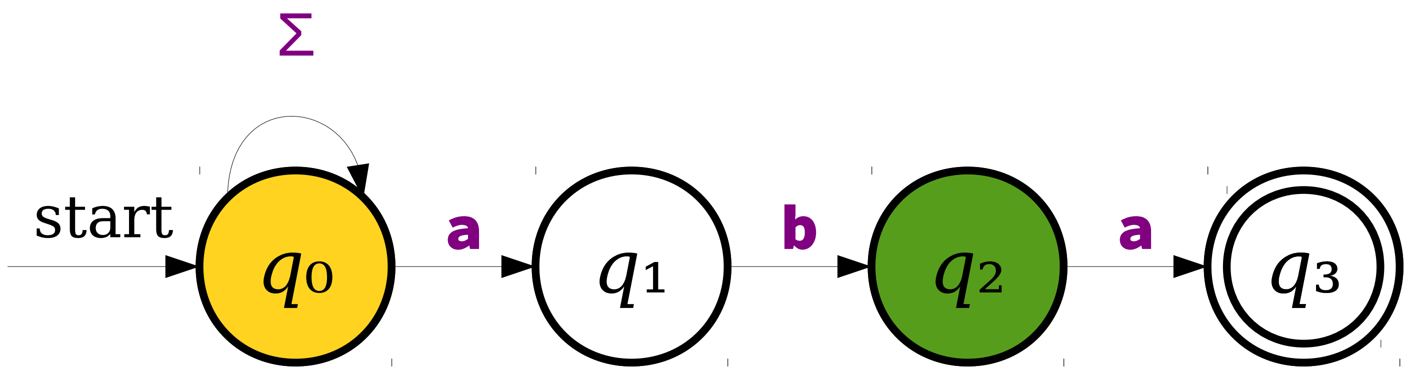

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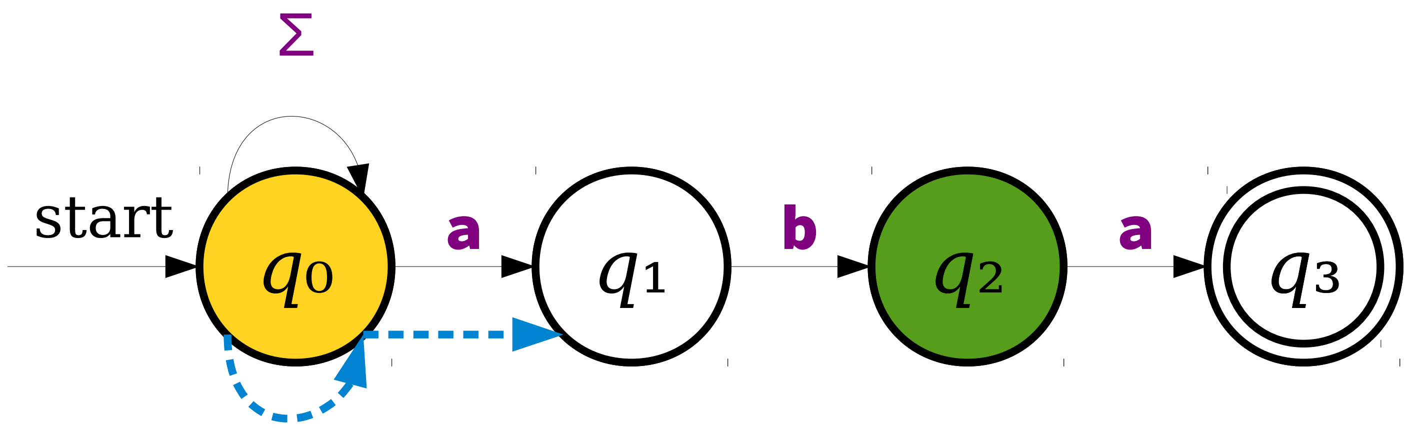

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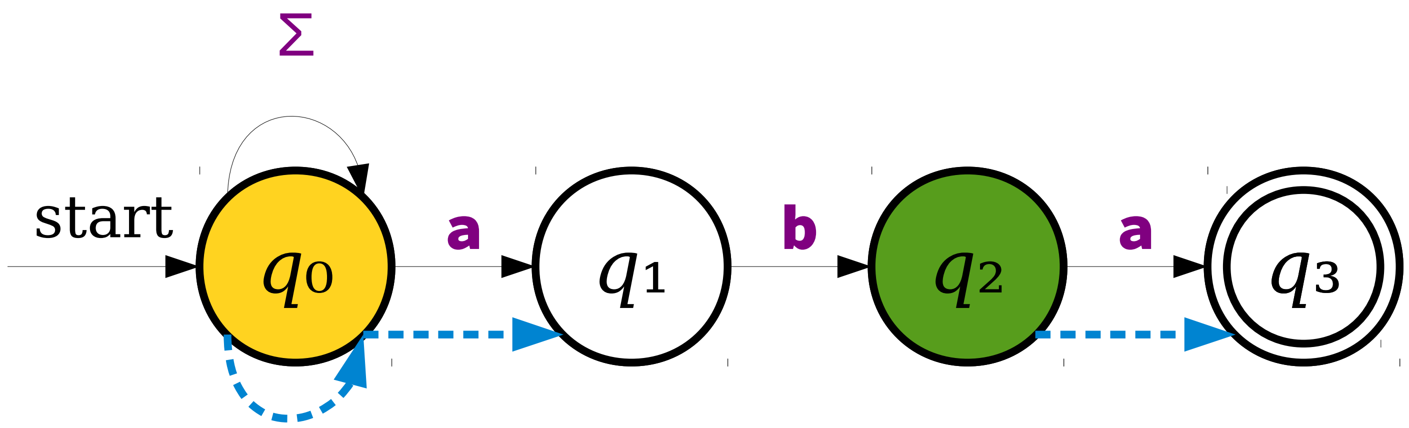

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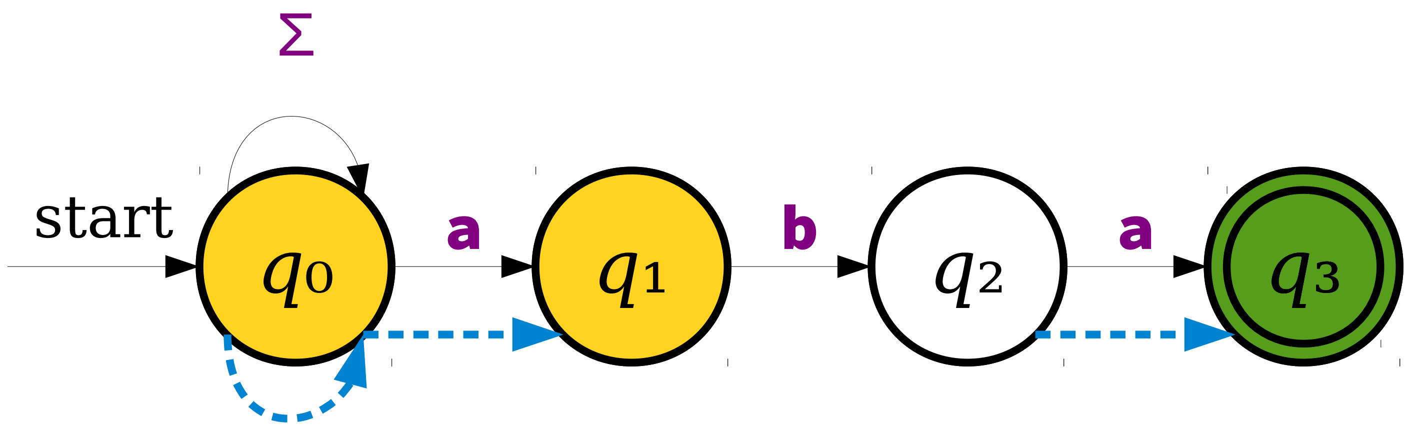

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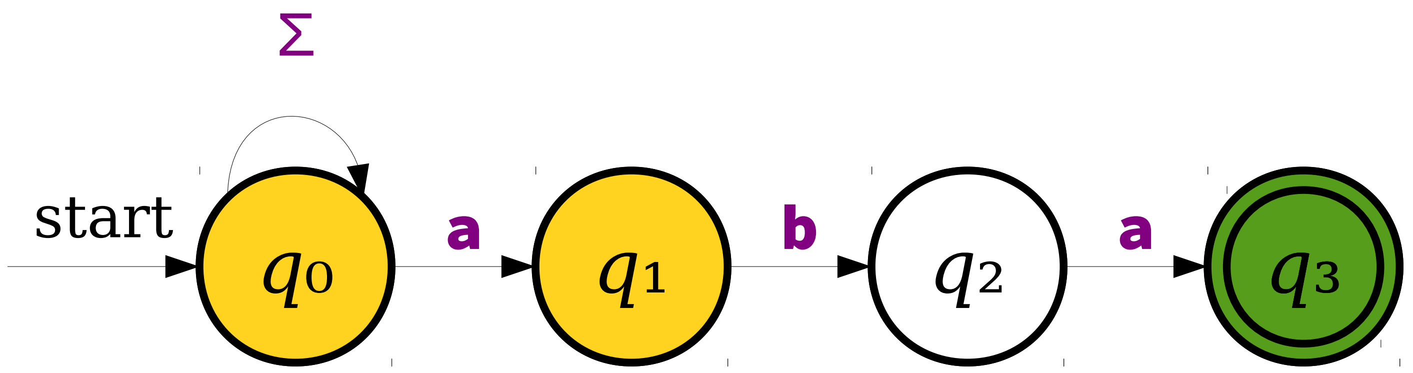

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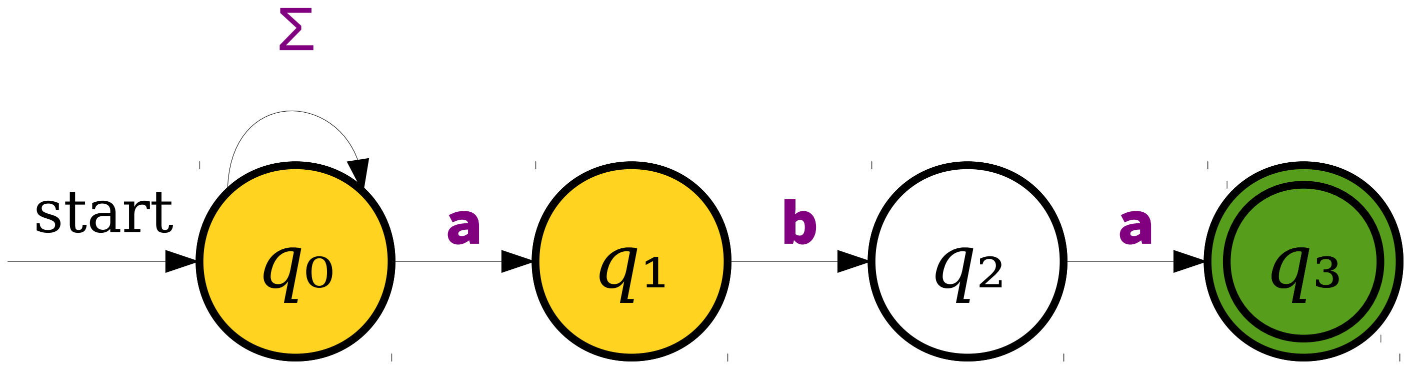

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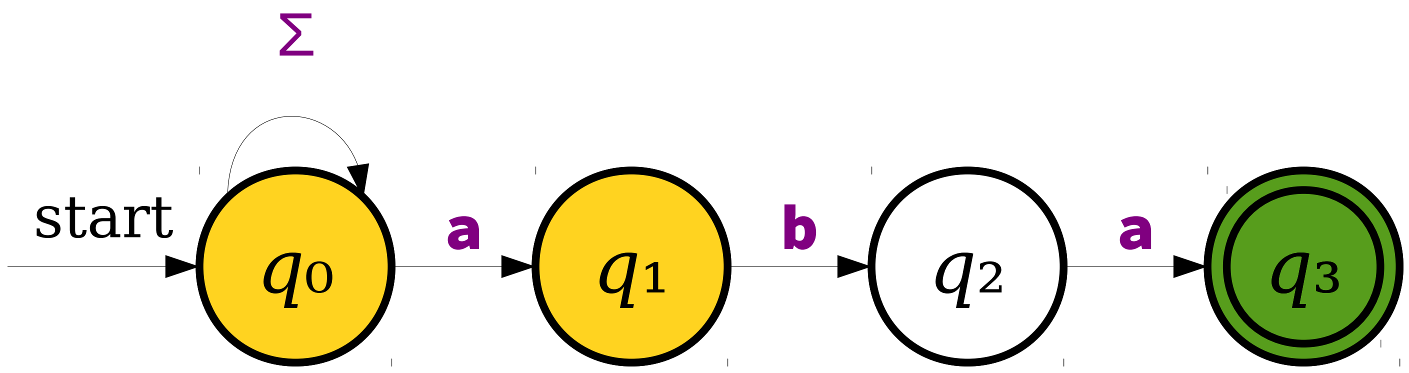
Your turn: What are the contents of the next row?



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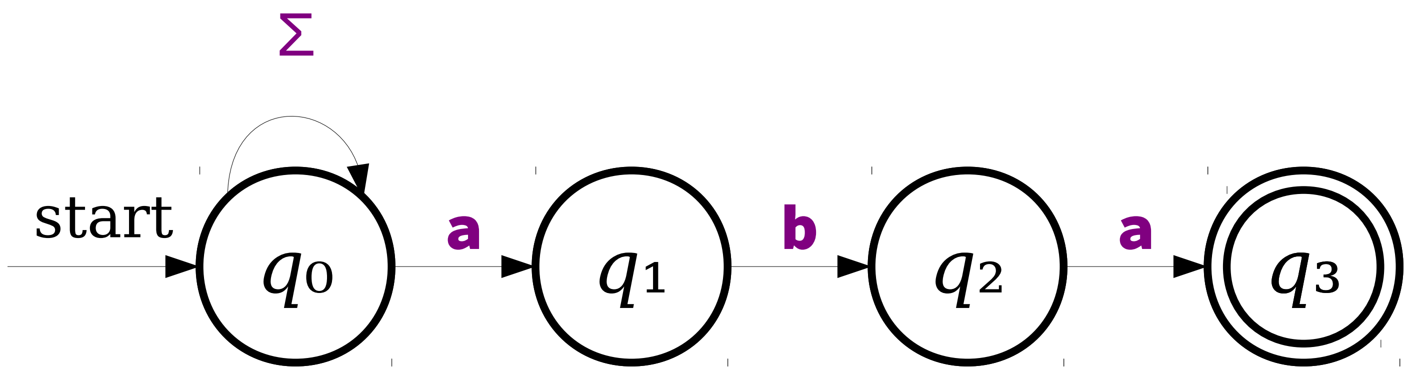

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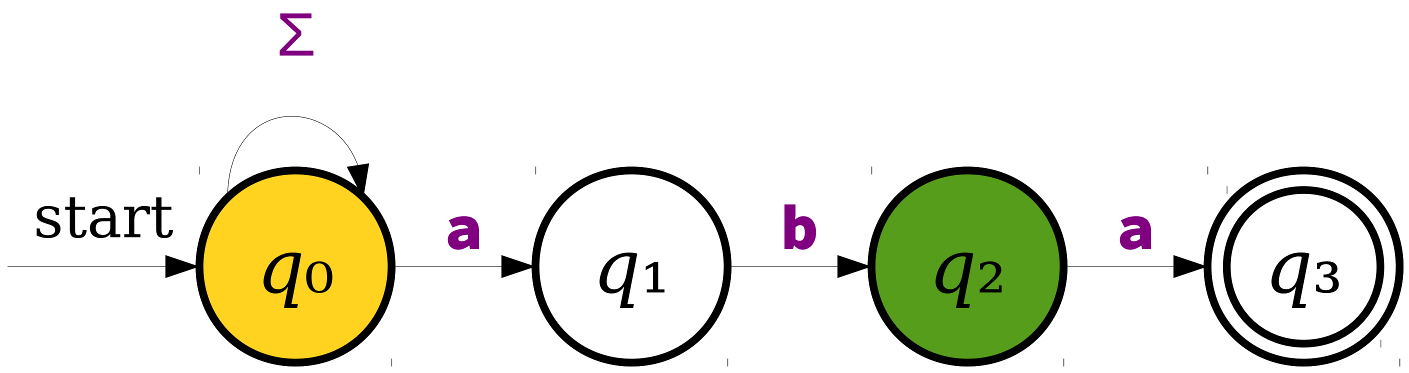

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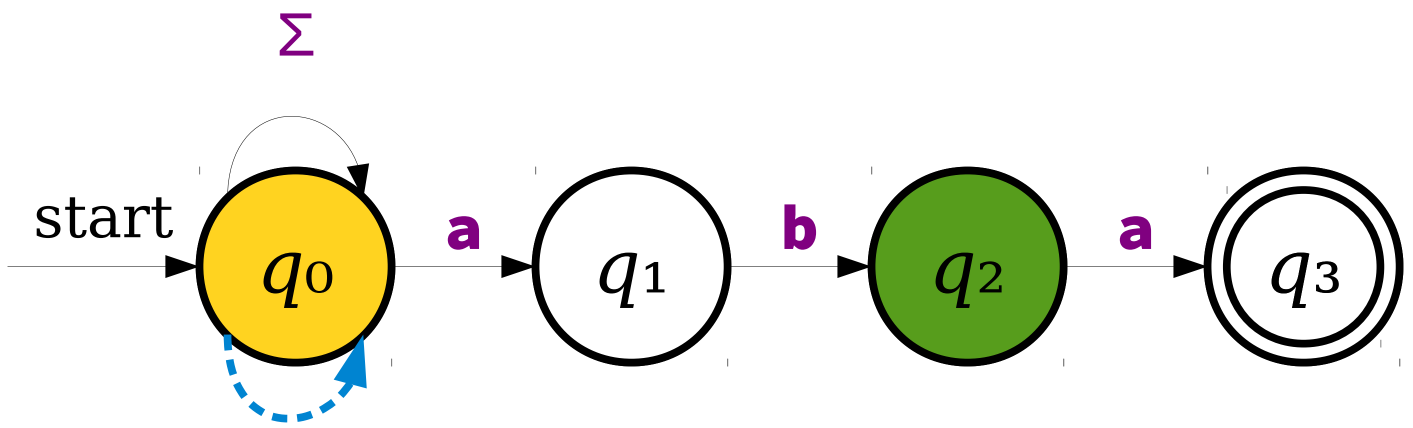

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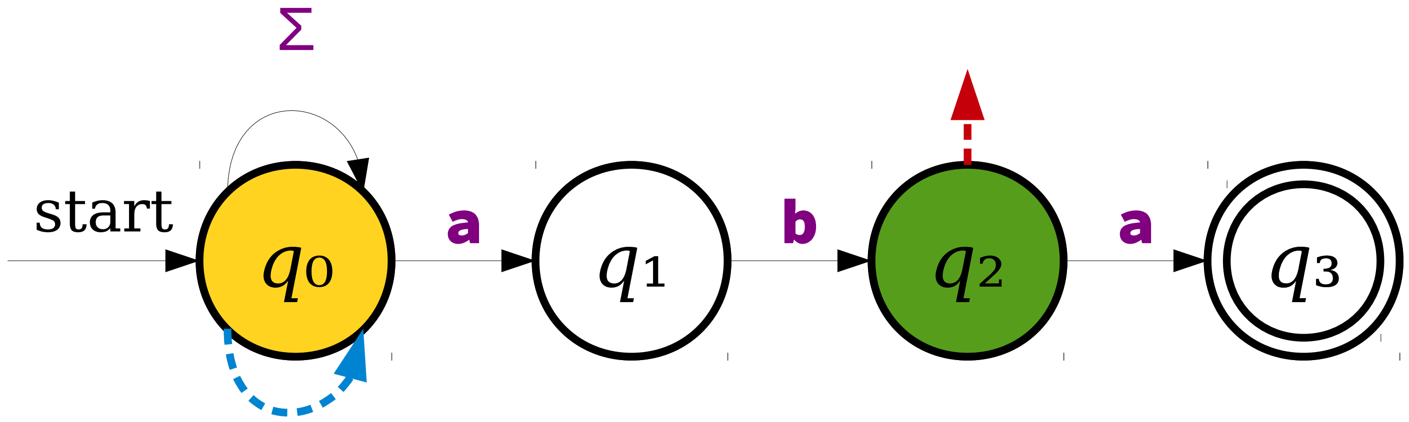

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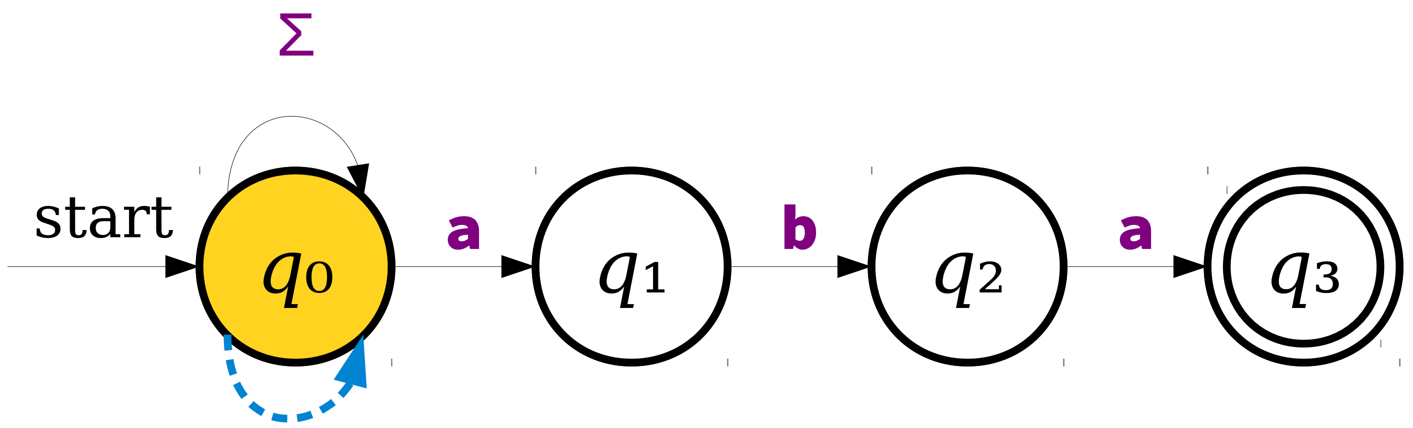

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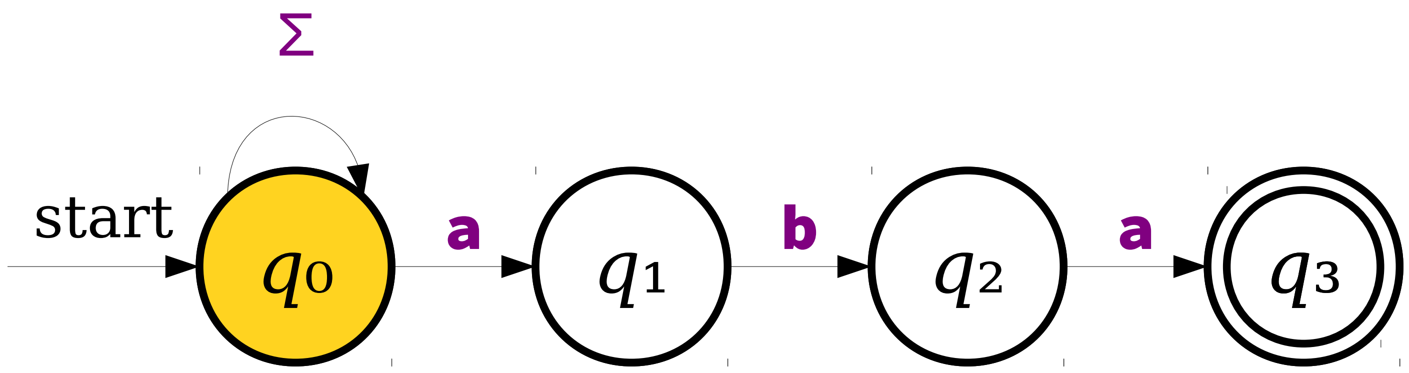

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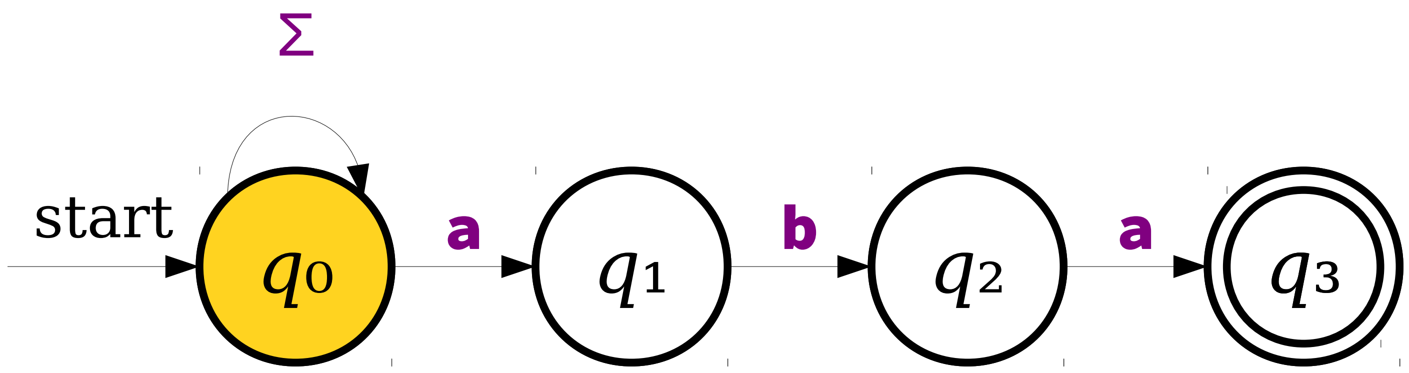

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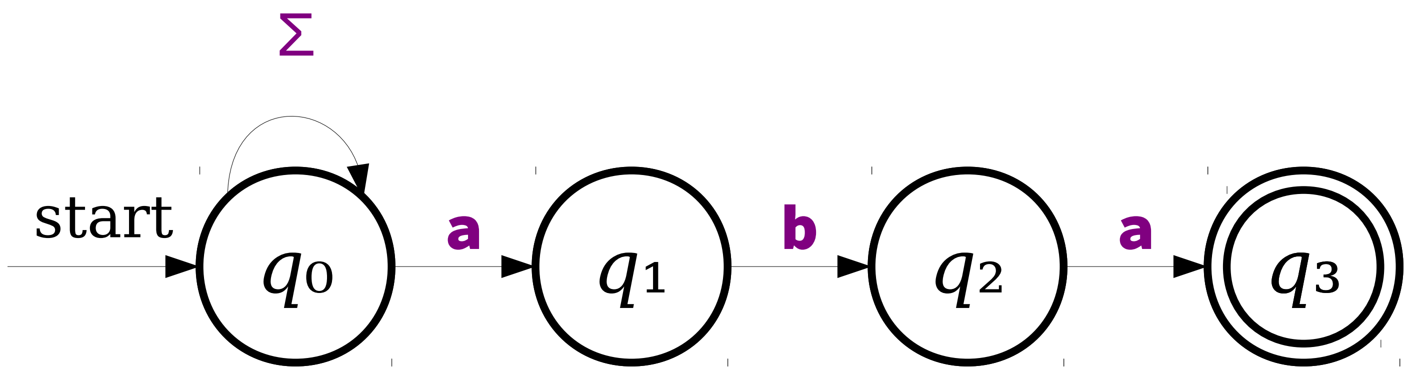

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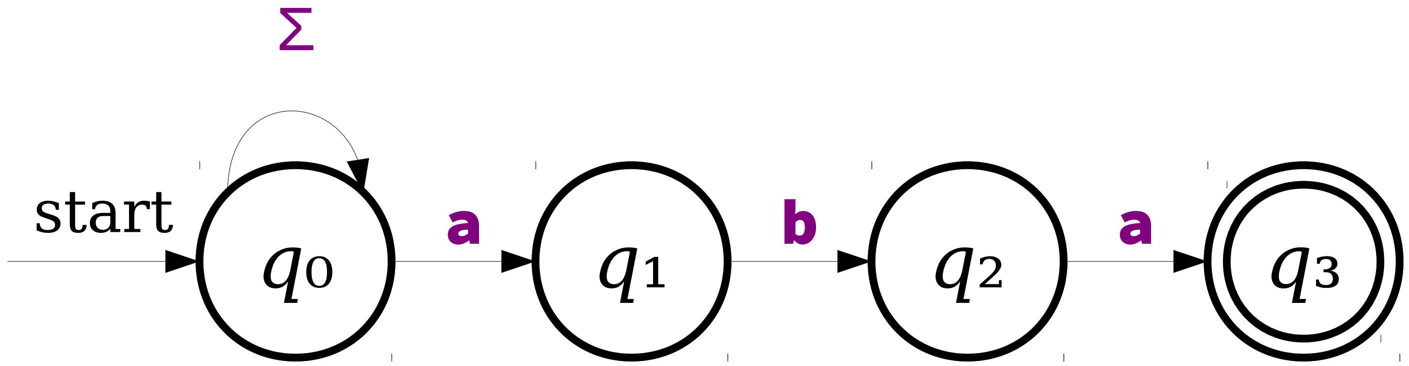

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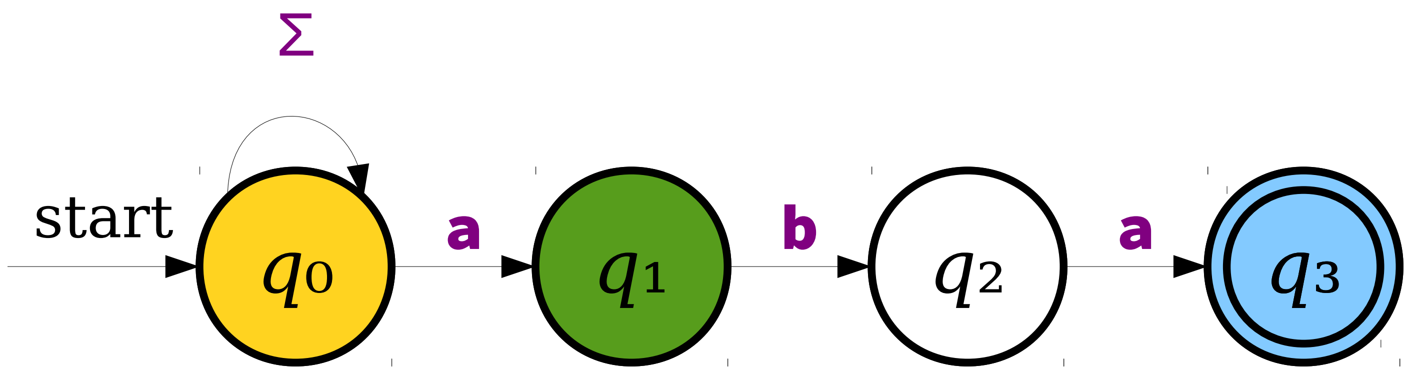

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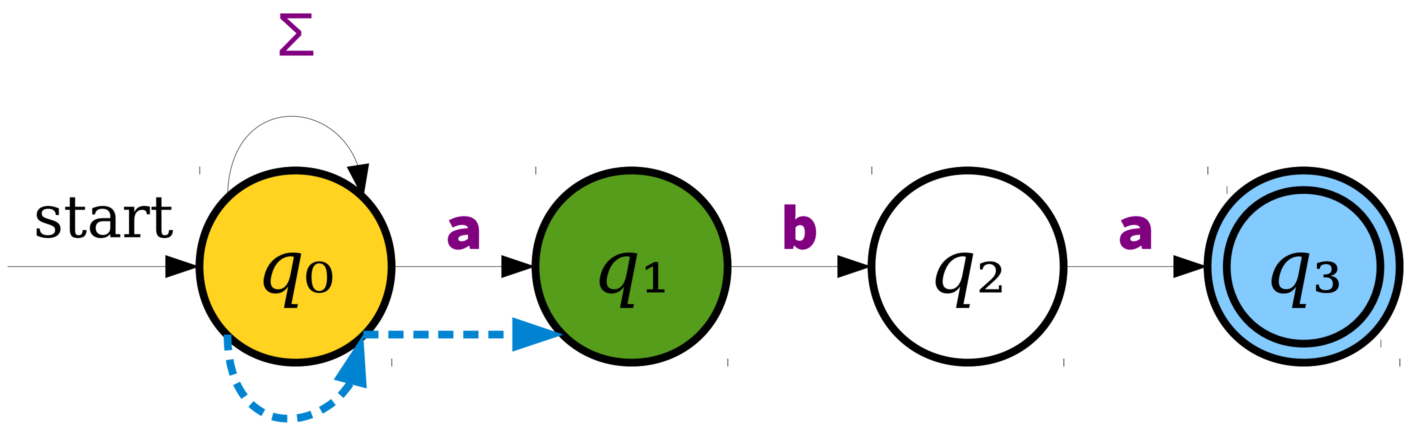

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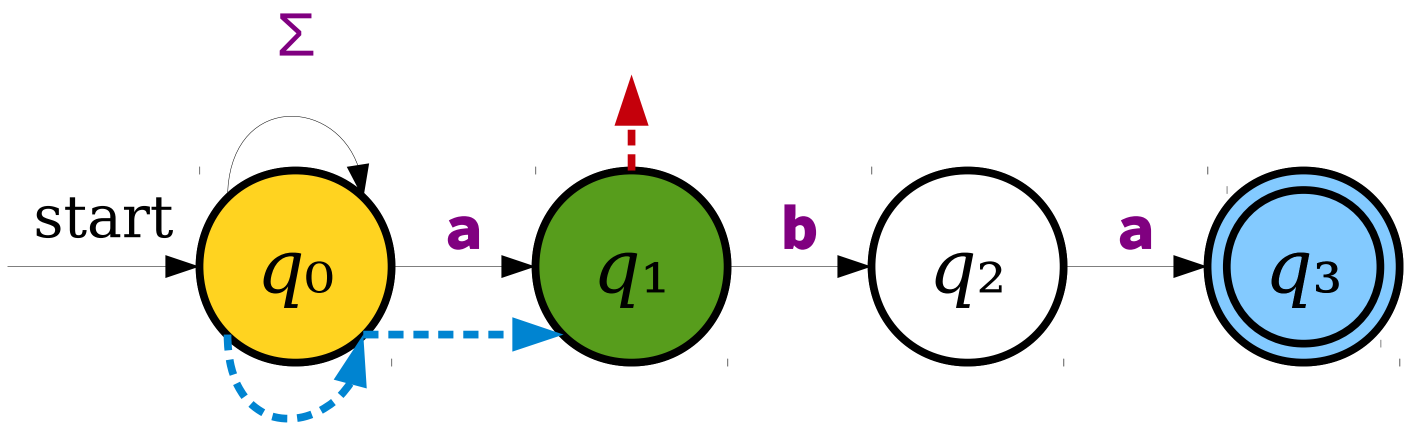

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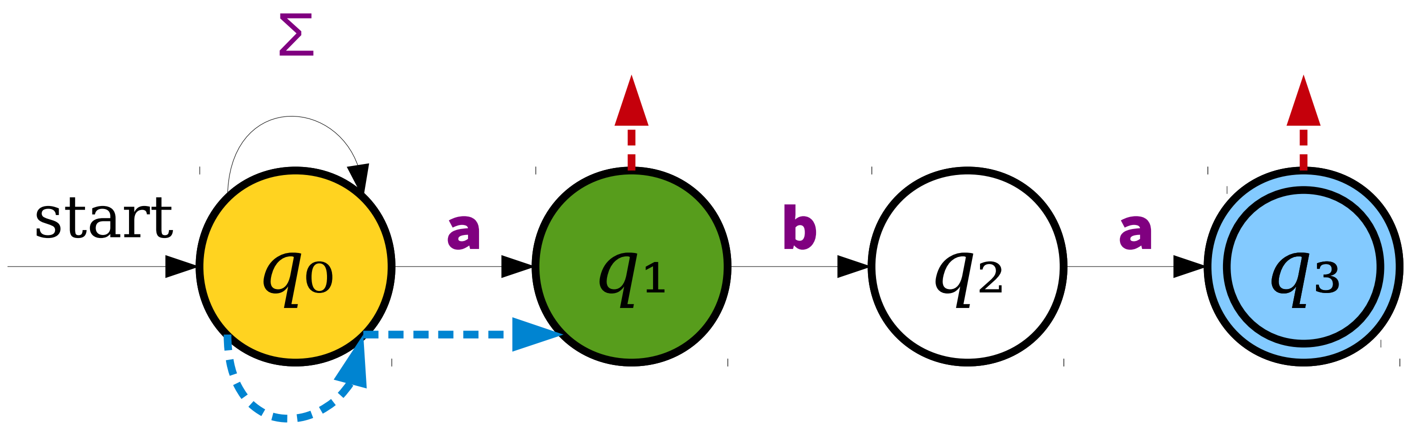

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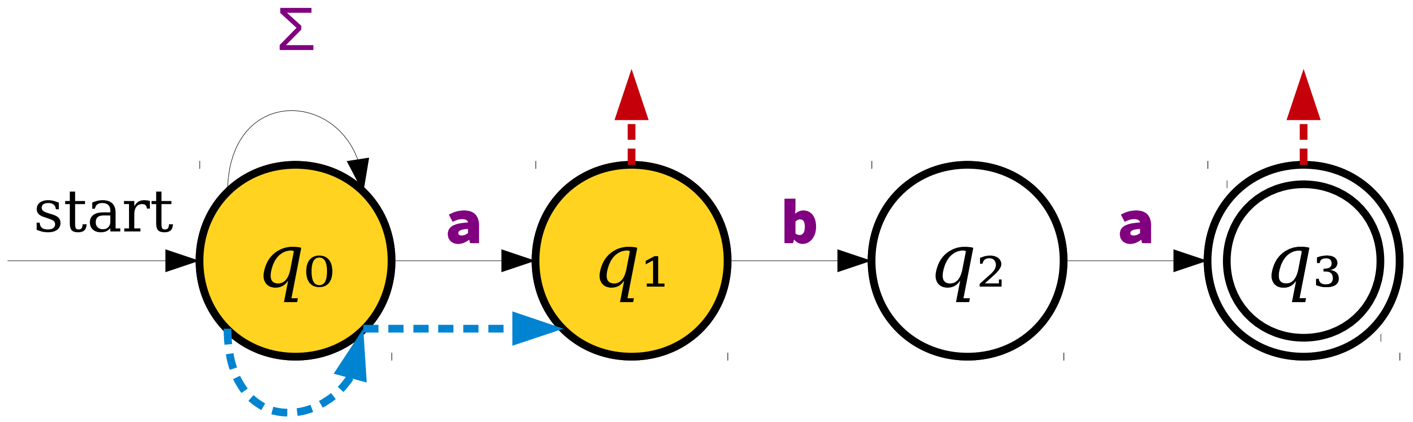

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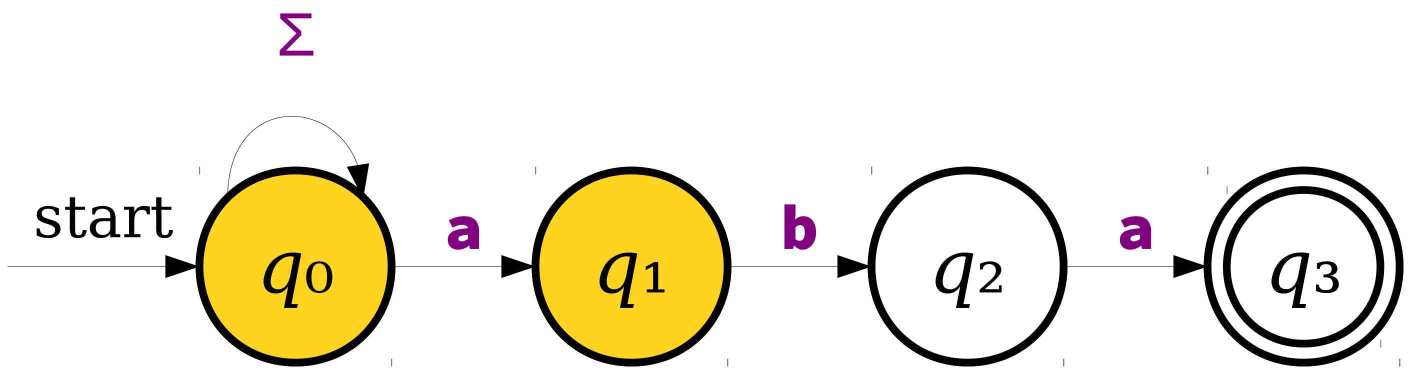

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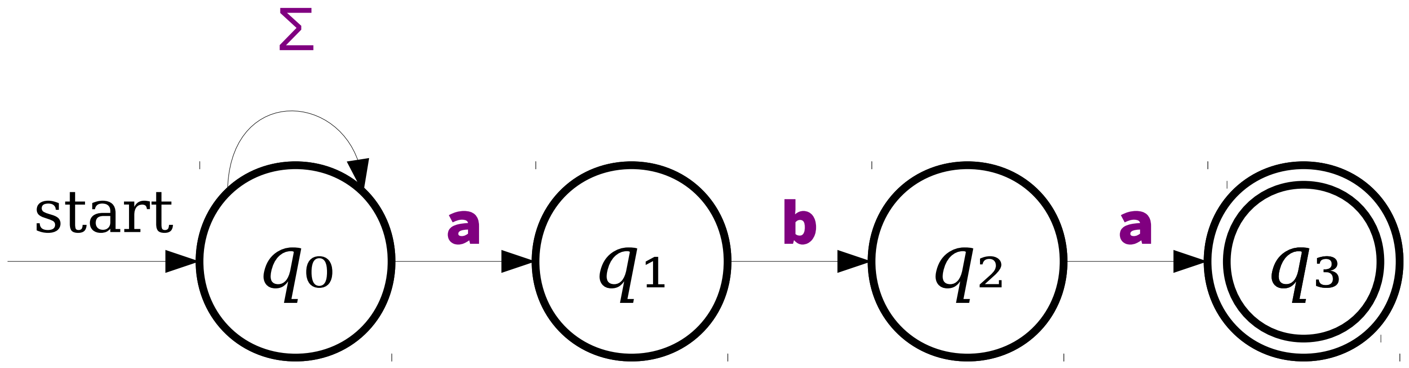

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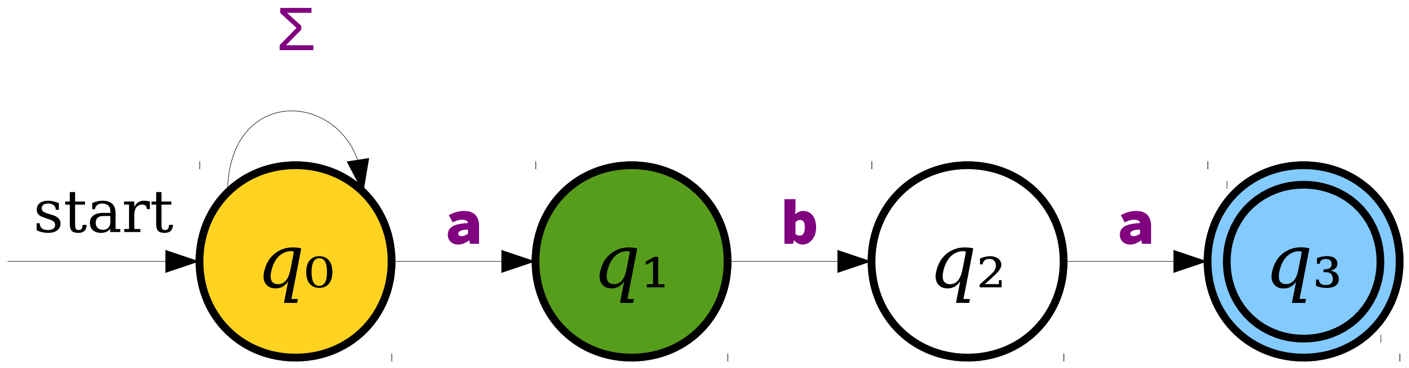

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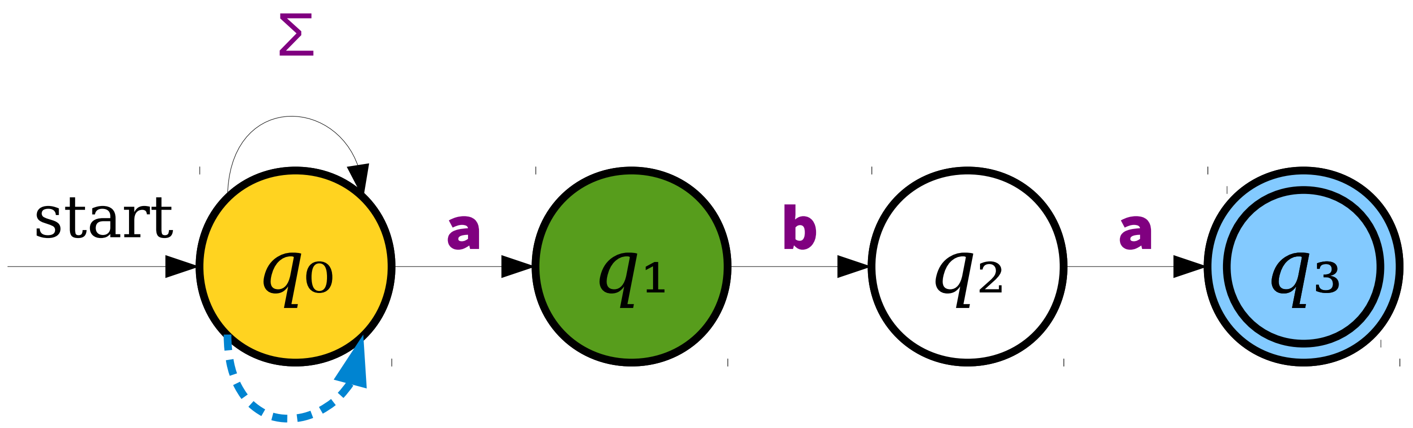

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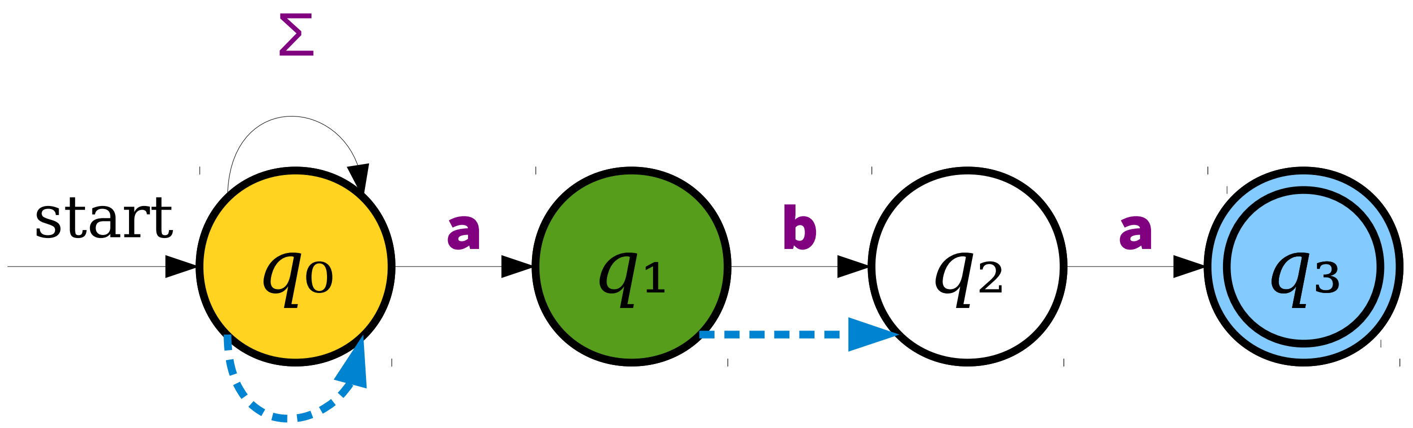

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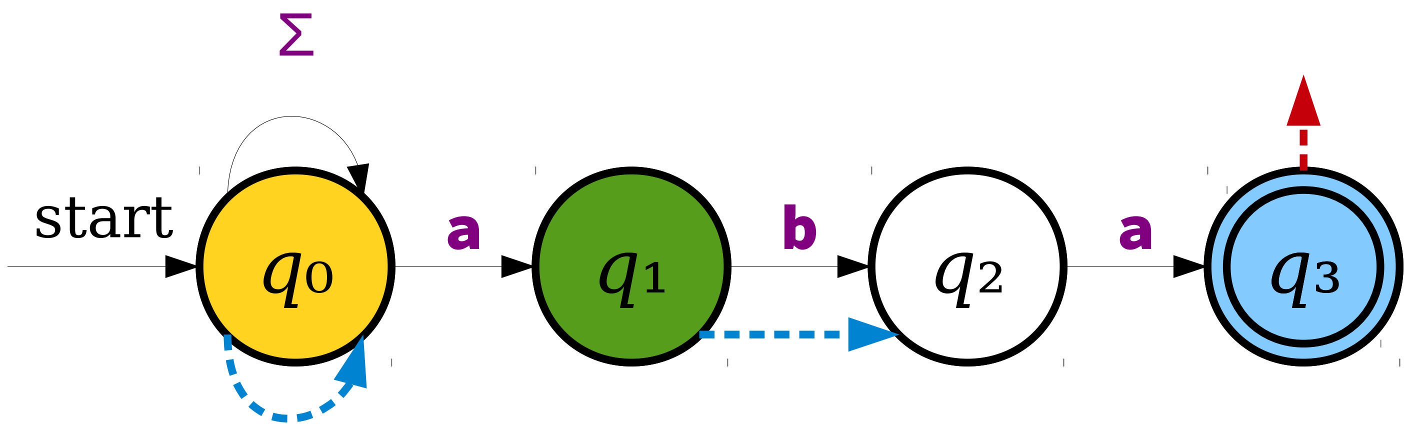

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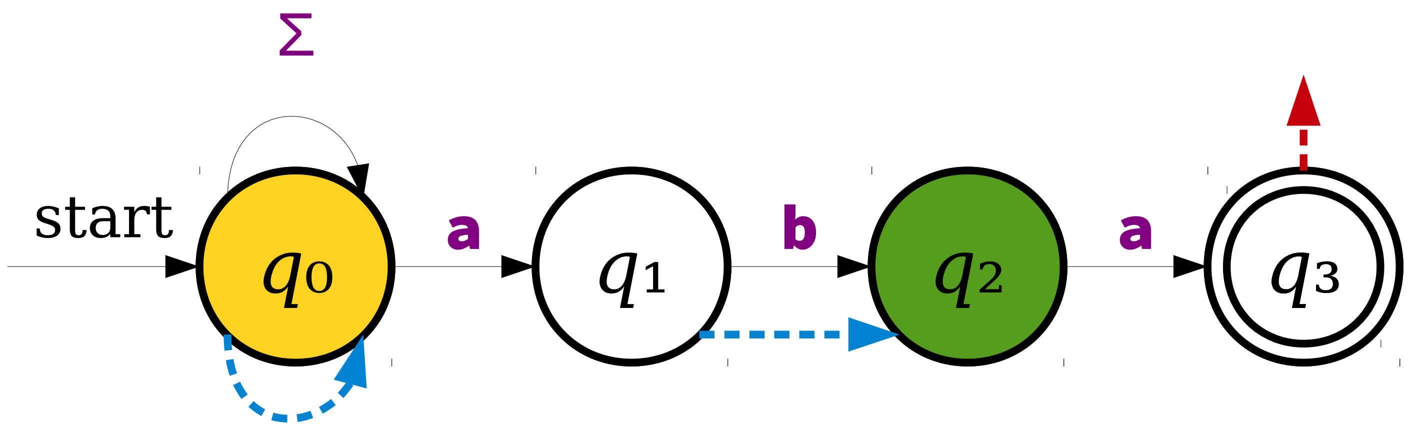

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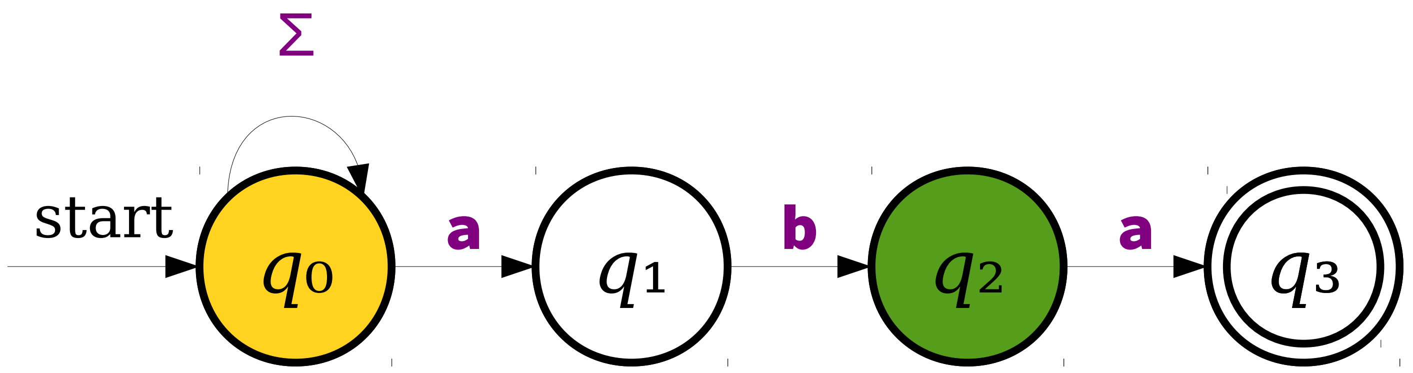

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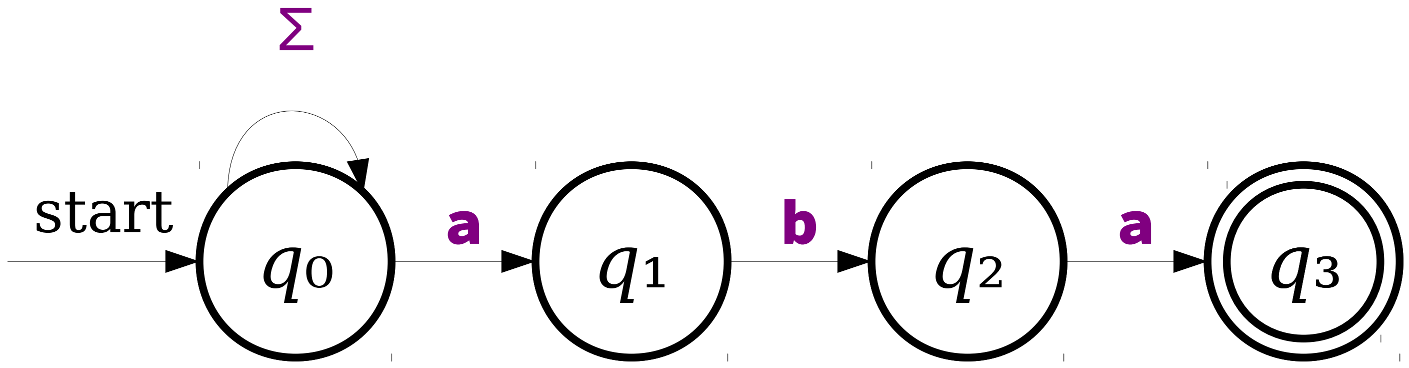

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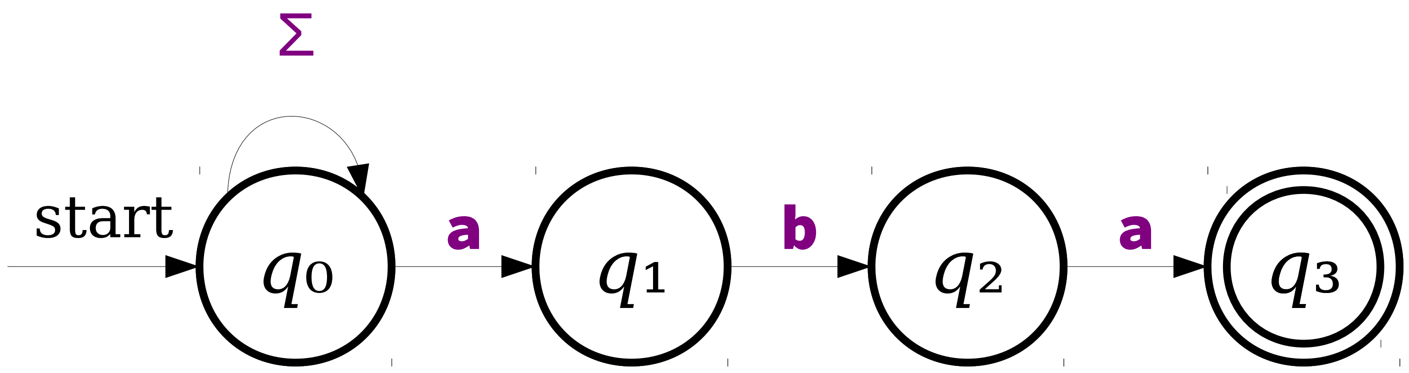

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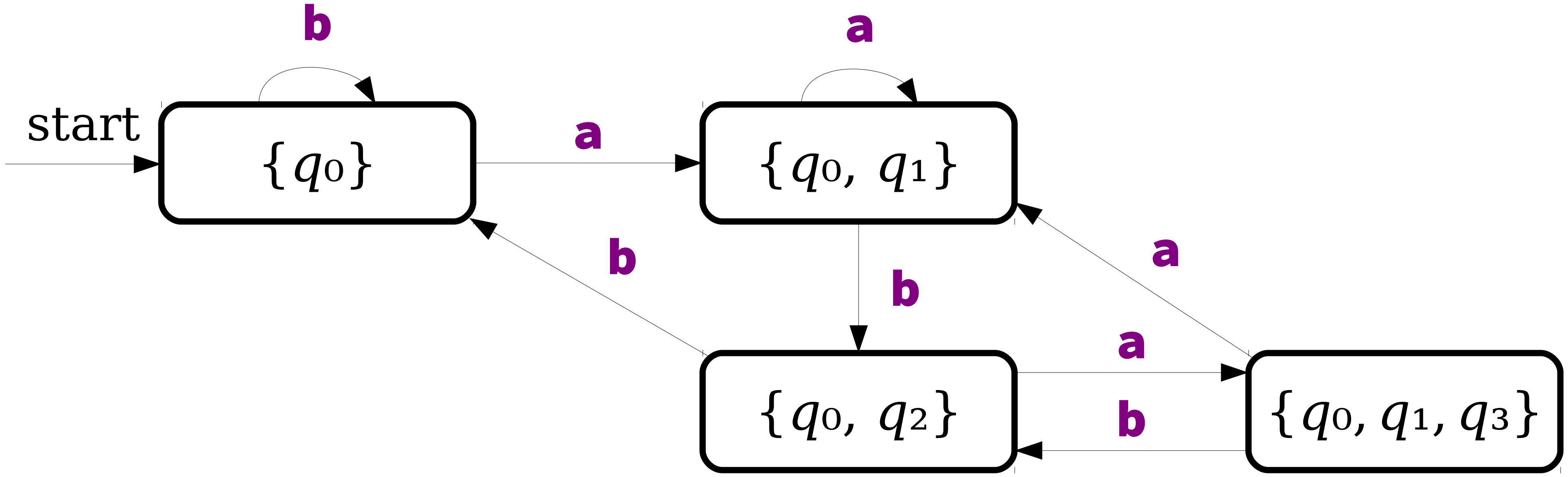

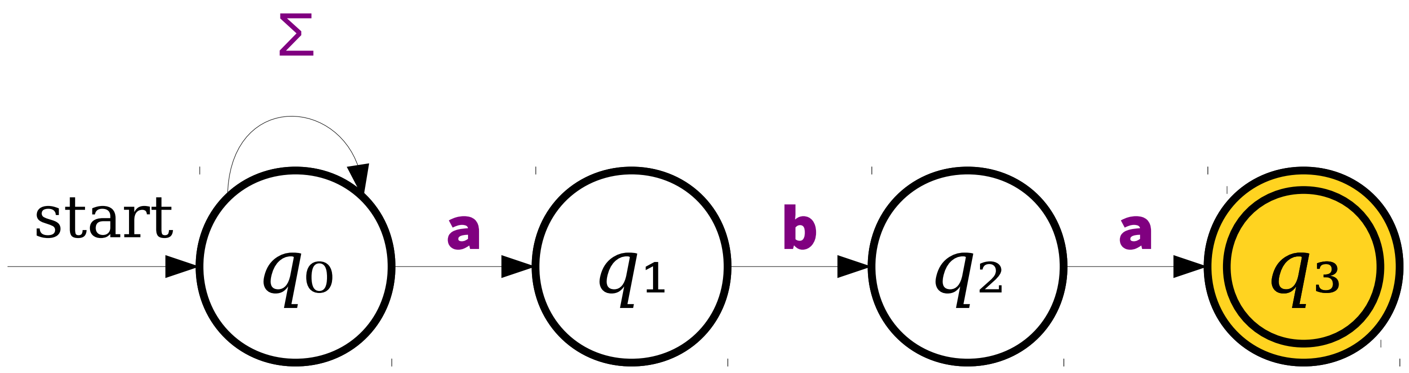

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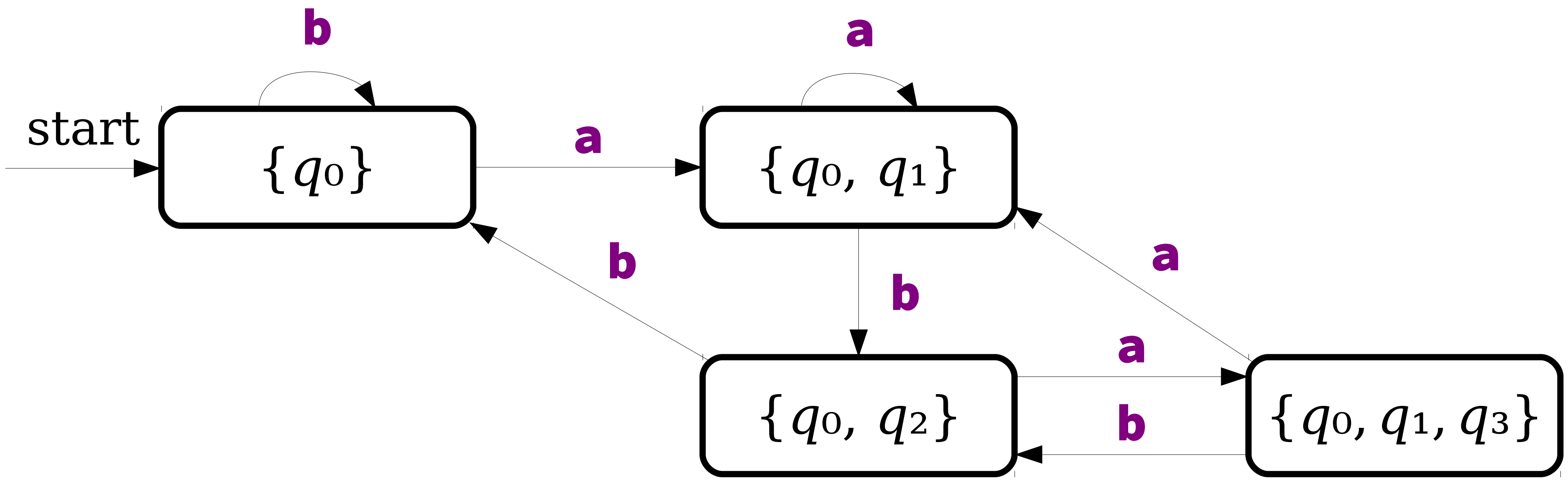

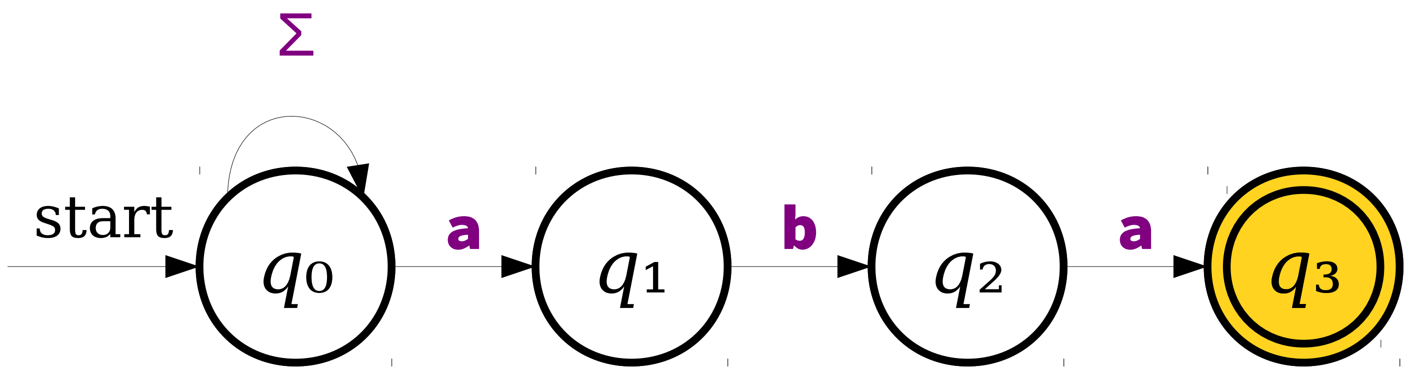

	a	b
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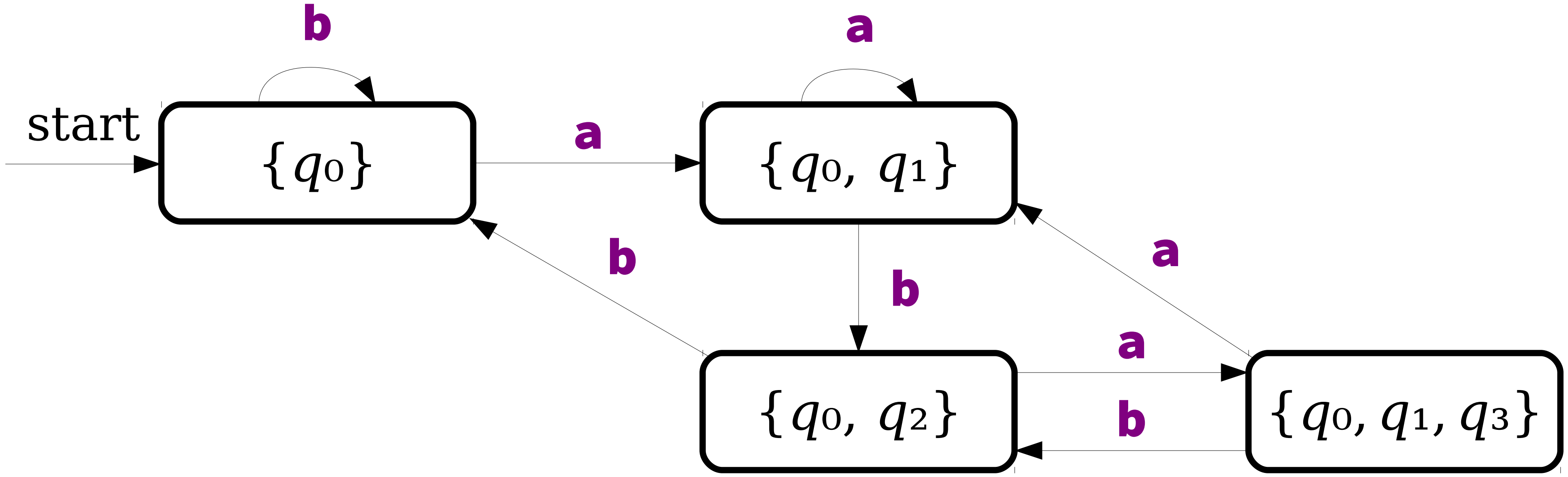
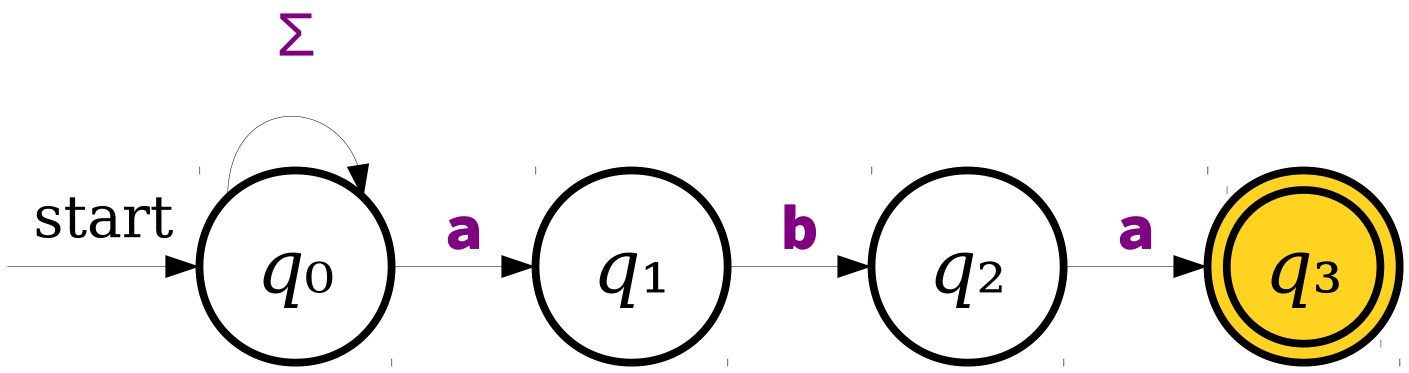
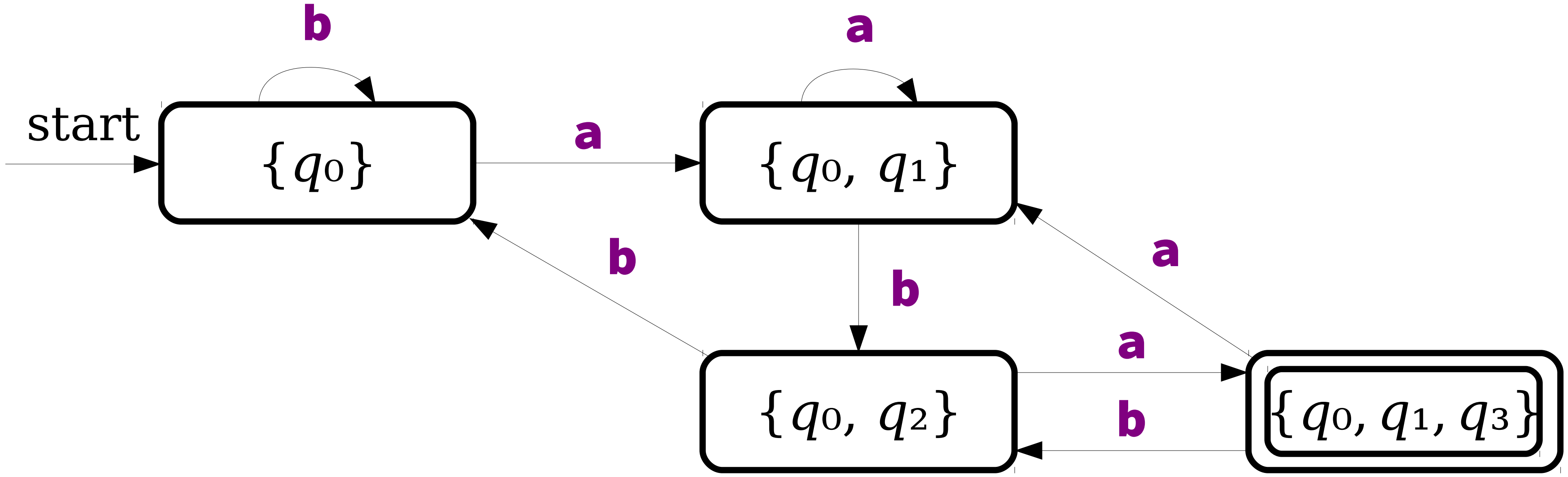
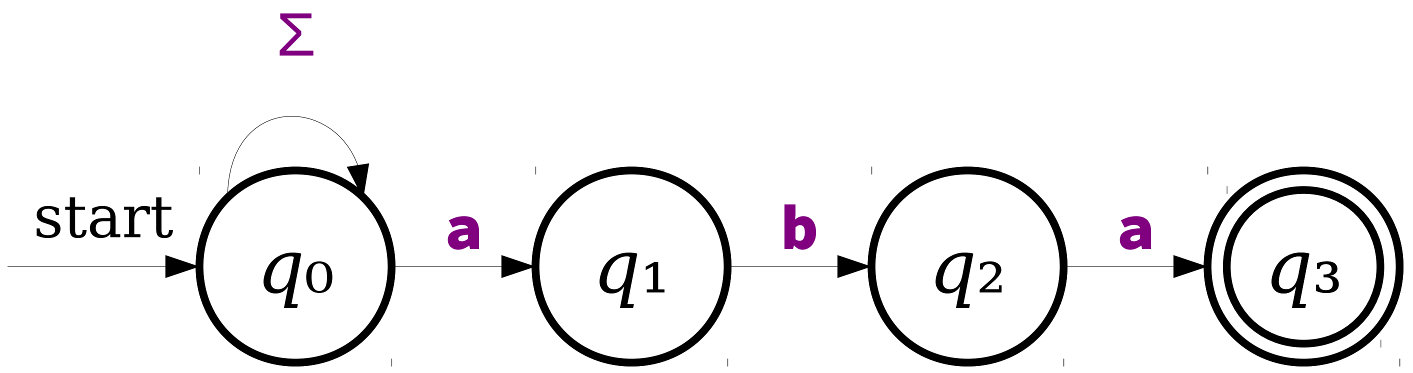


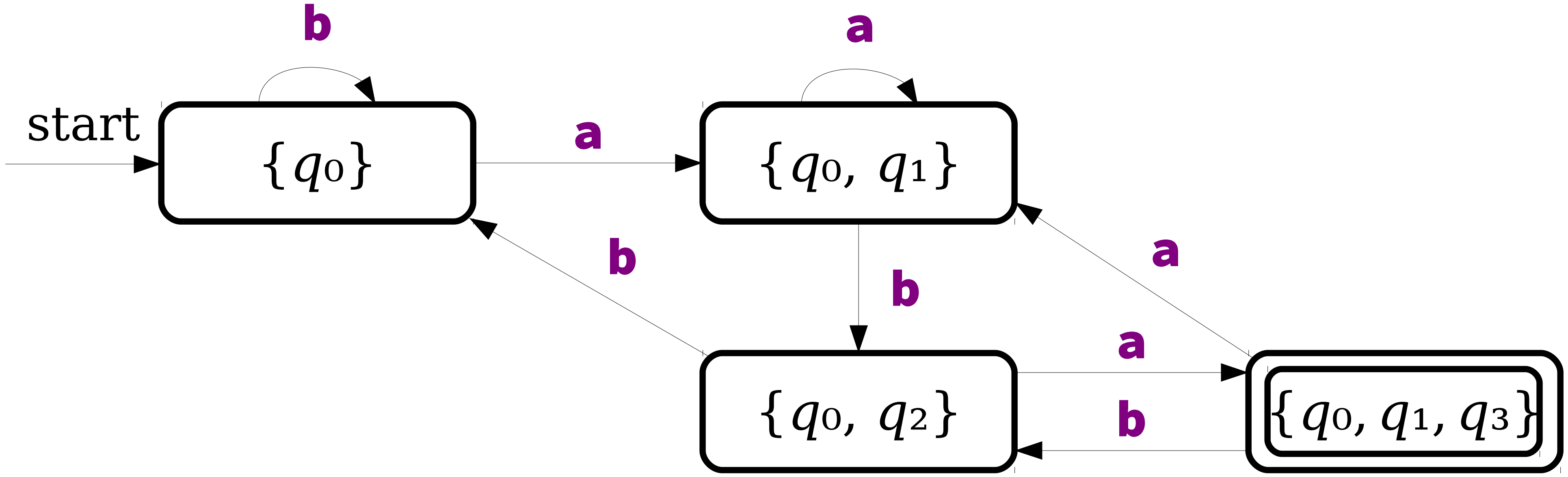


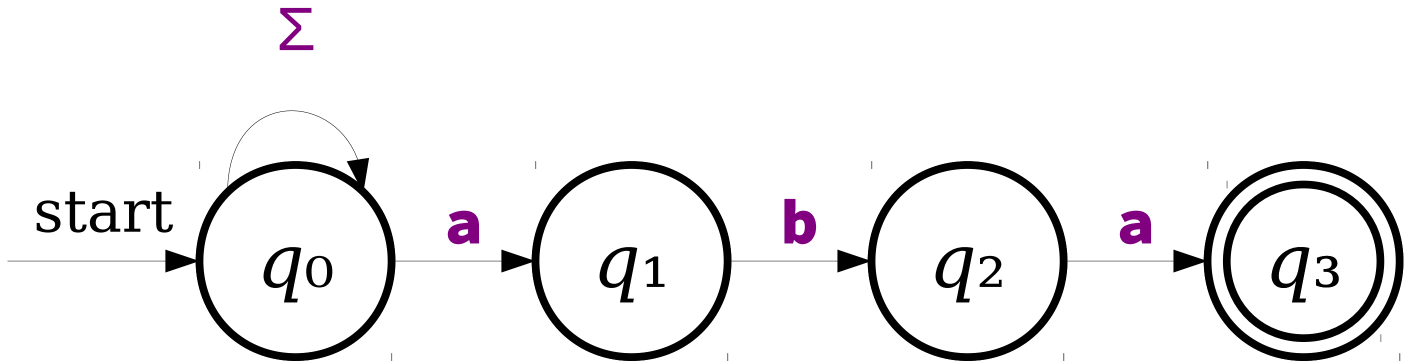
	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$*\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$





	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$*\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$




Σ
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q_0
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q_1
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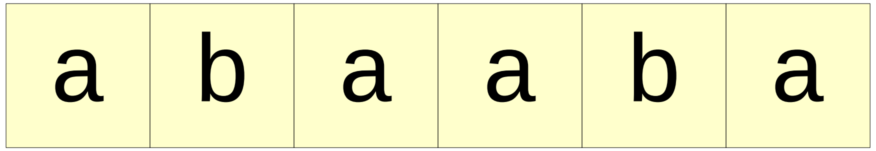
a
q_3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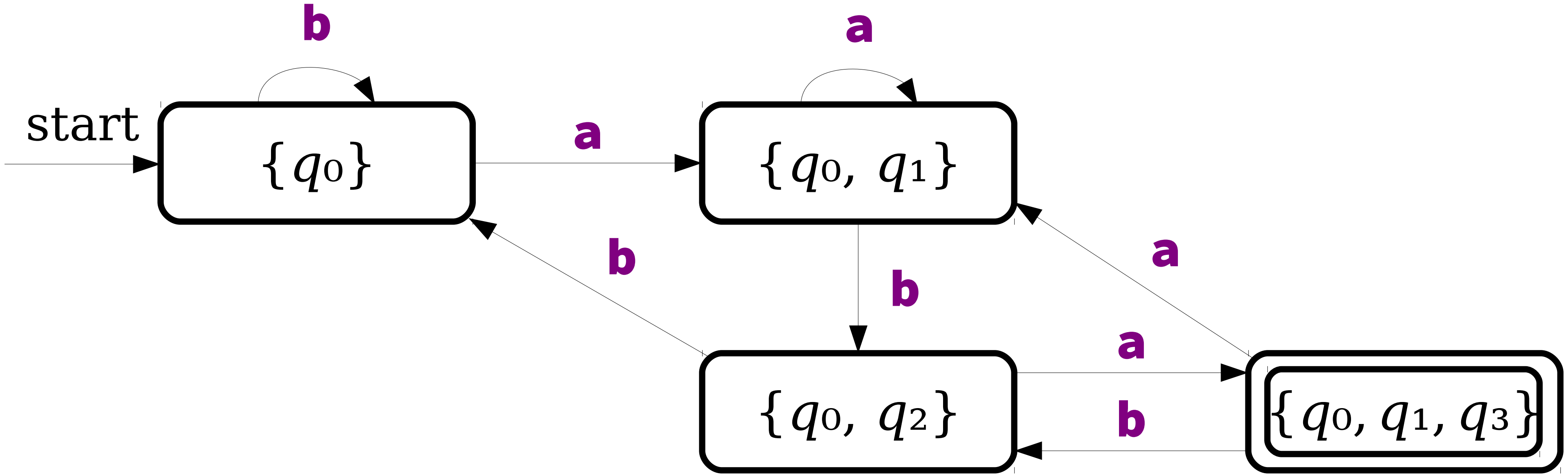
b
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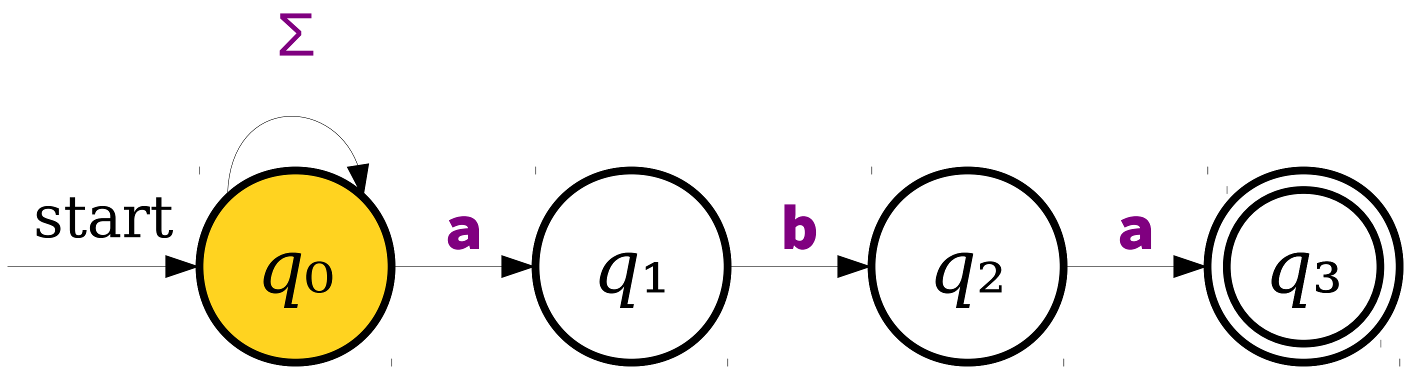
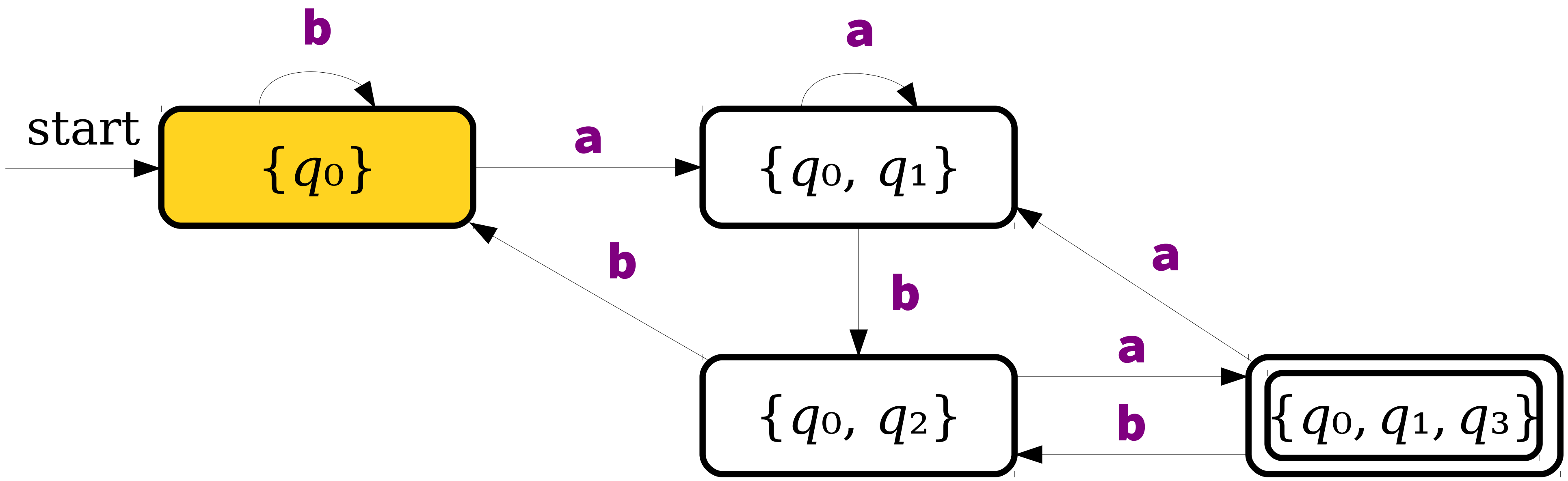
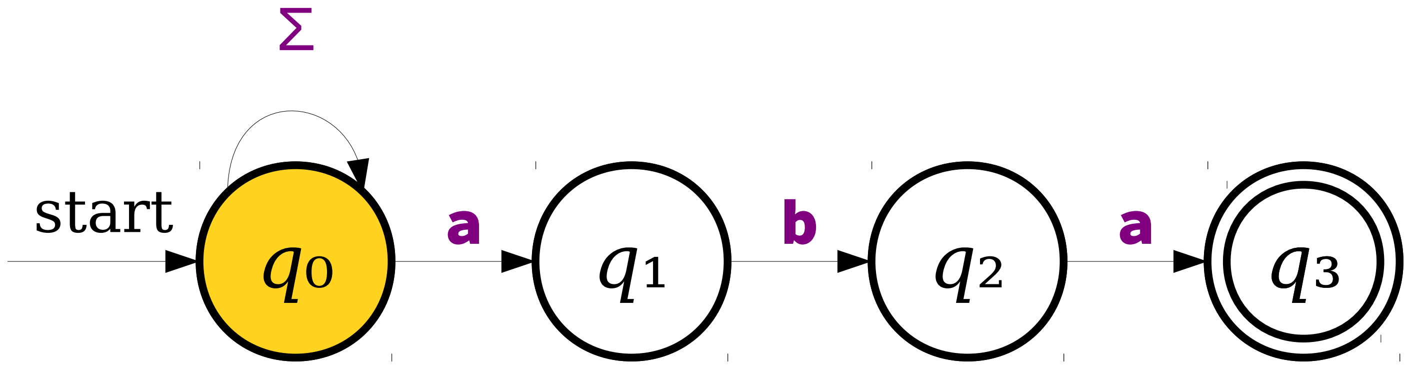


a	b	a	a	b	a

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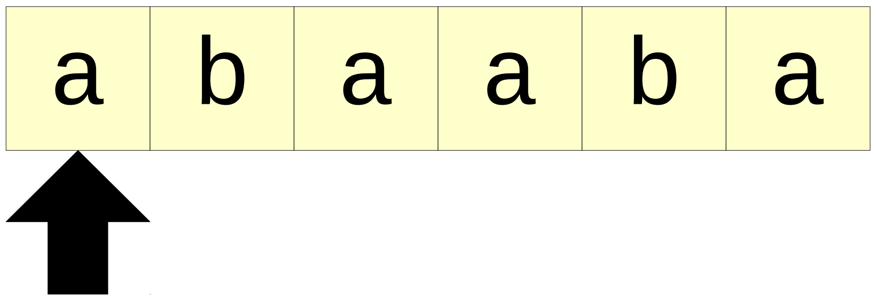
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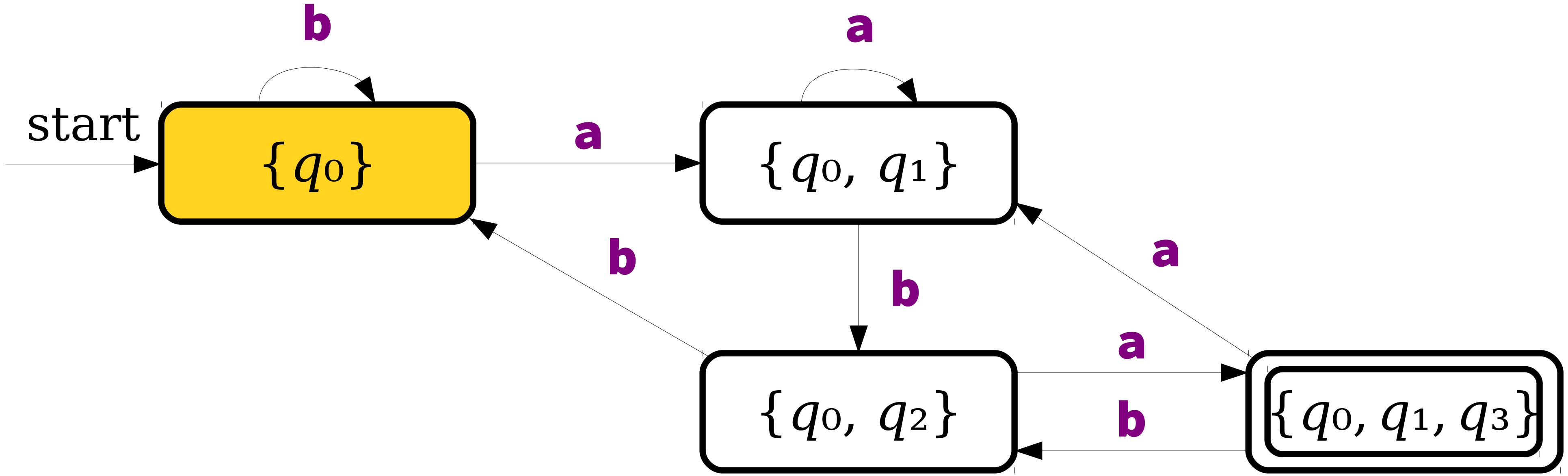
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$\{q_0, q_2\}$
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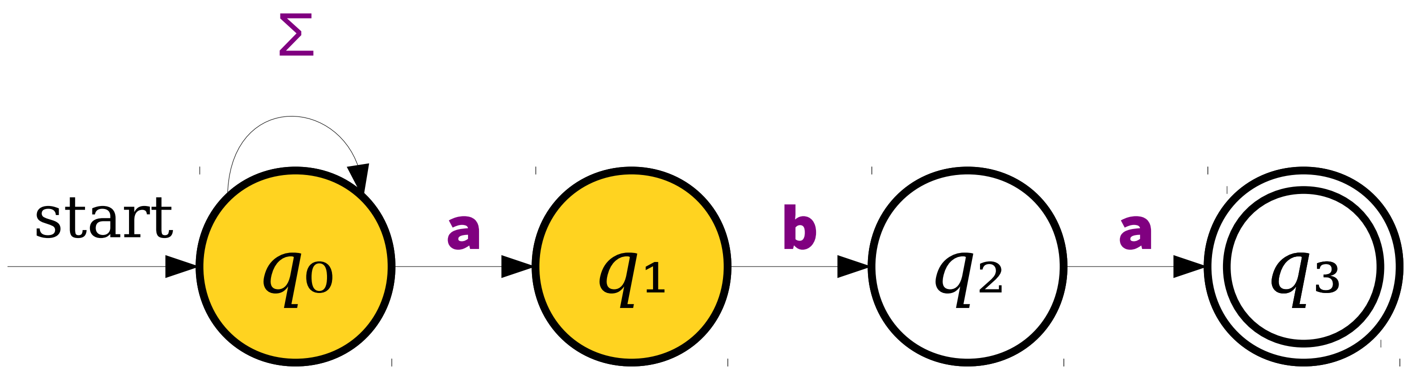
$\{q_0, q_1, q_3\}$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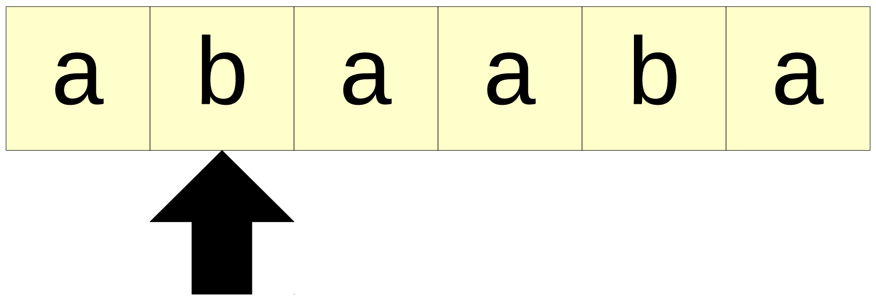
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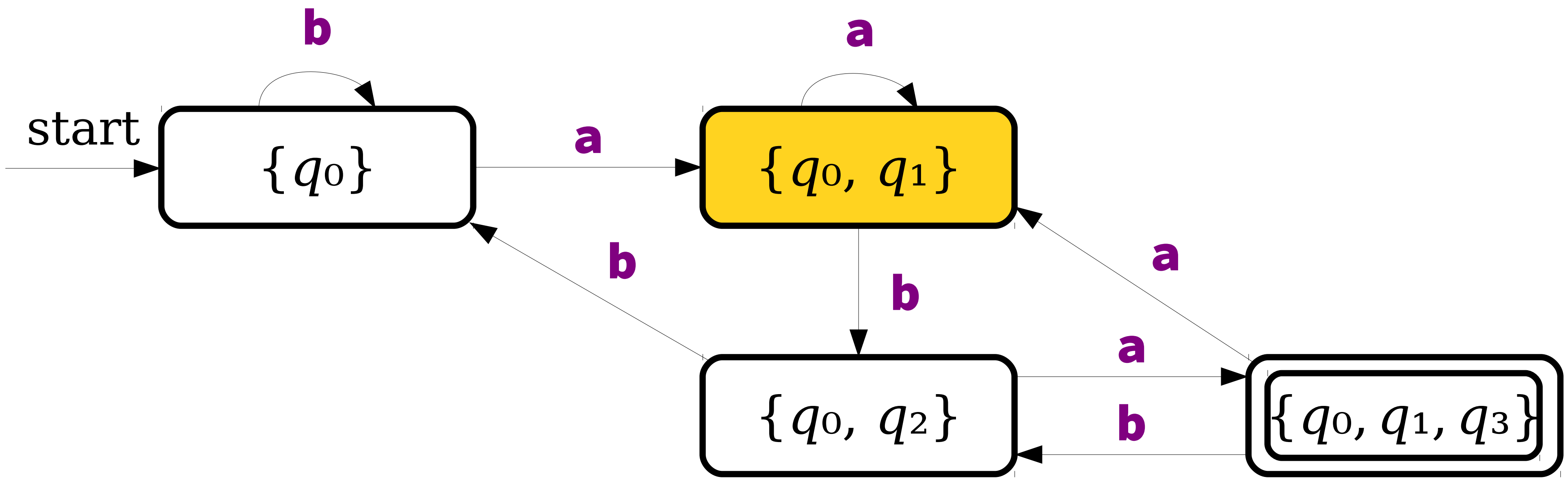
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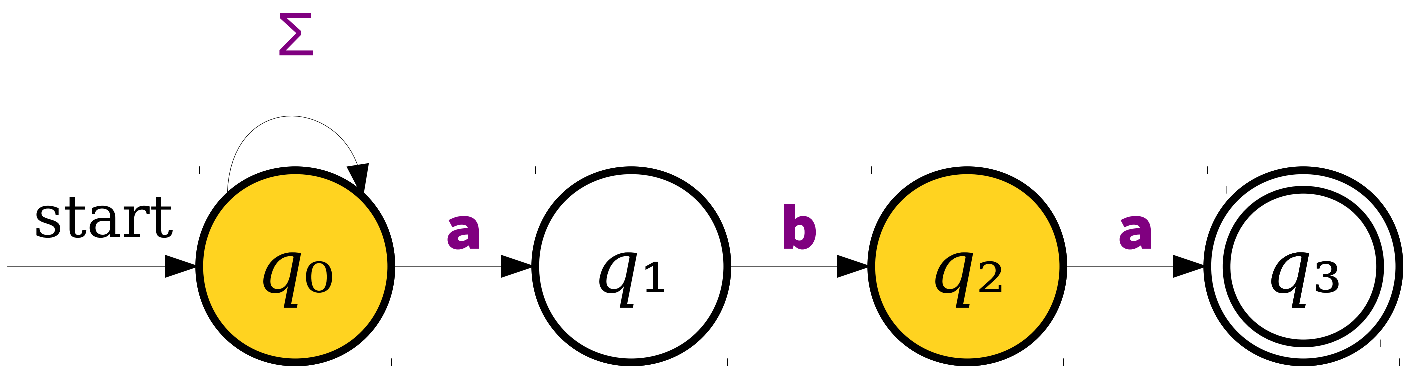
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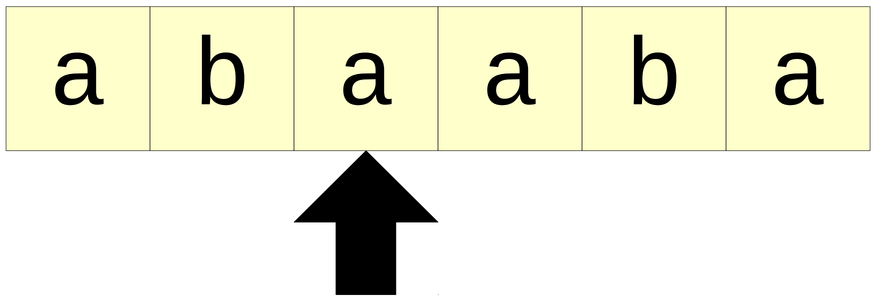
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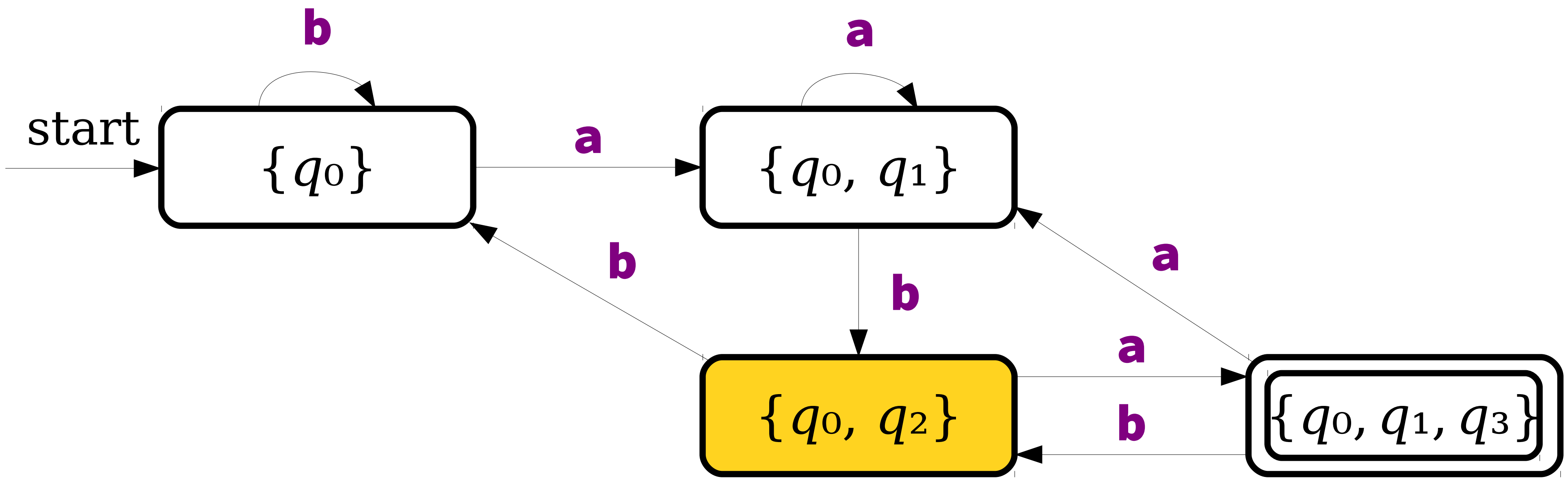
start
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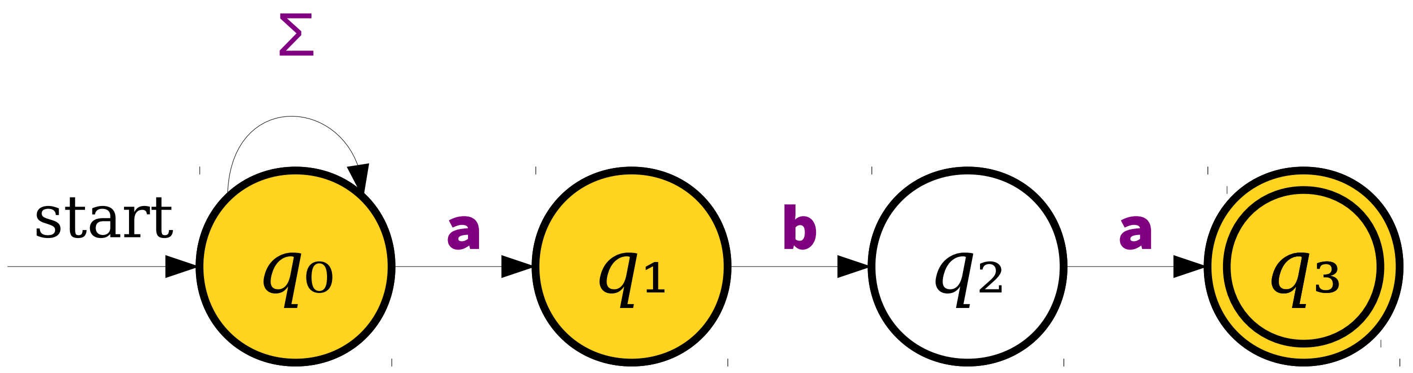

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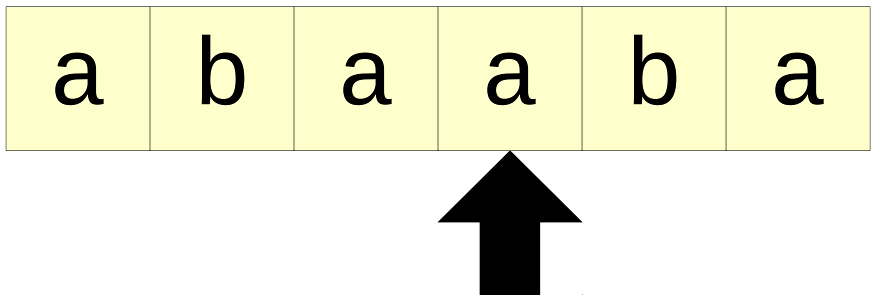

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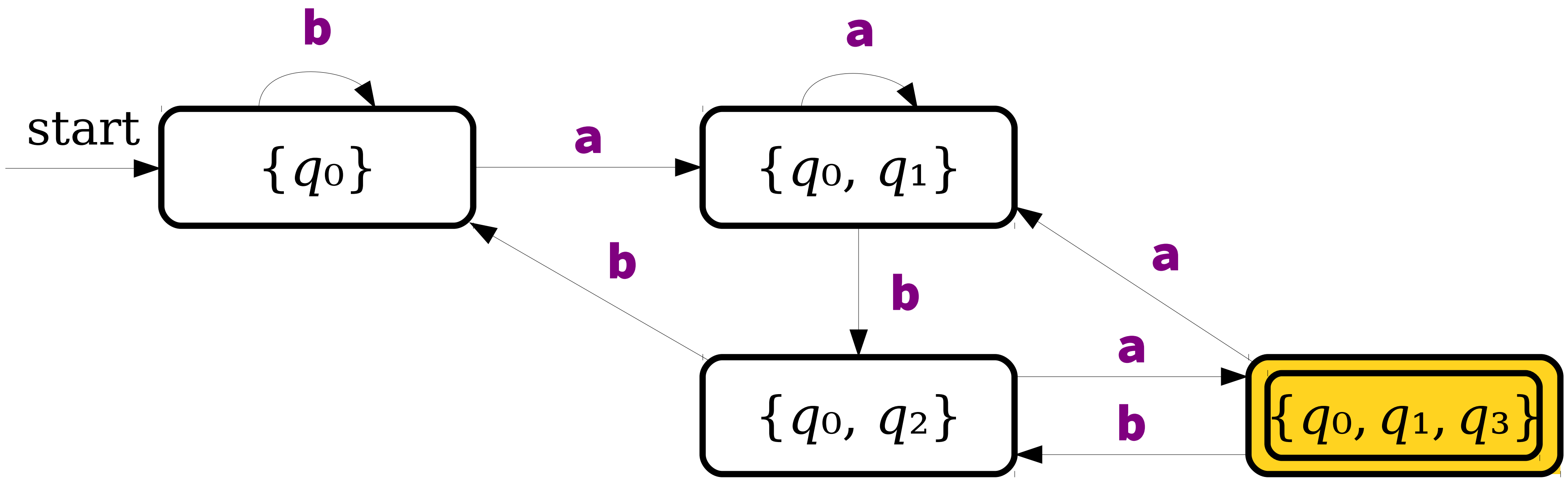

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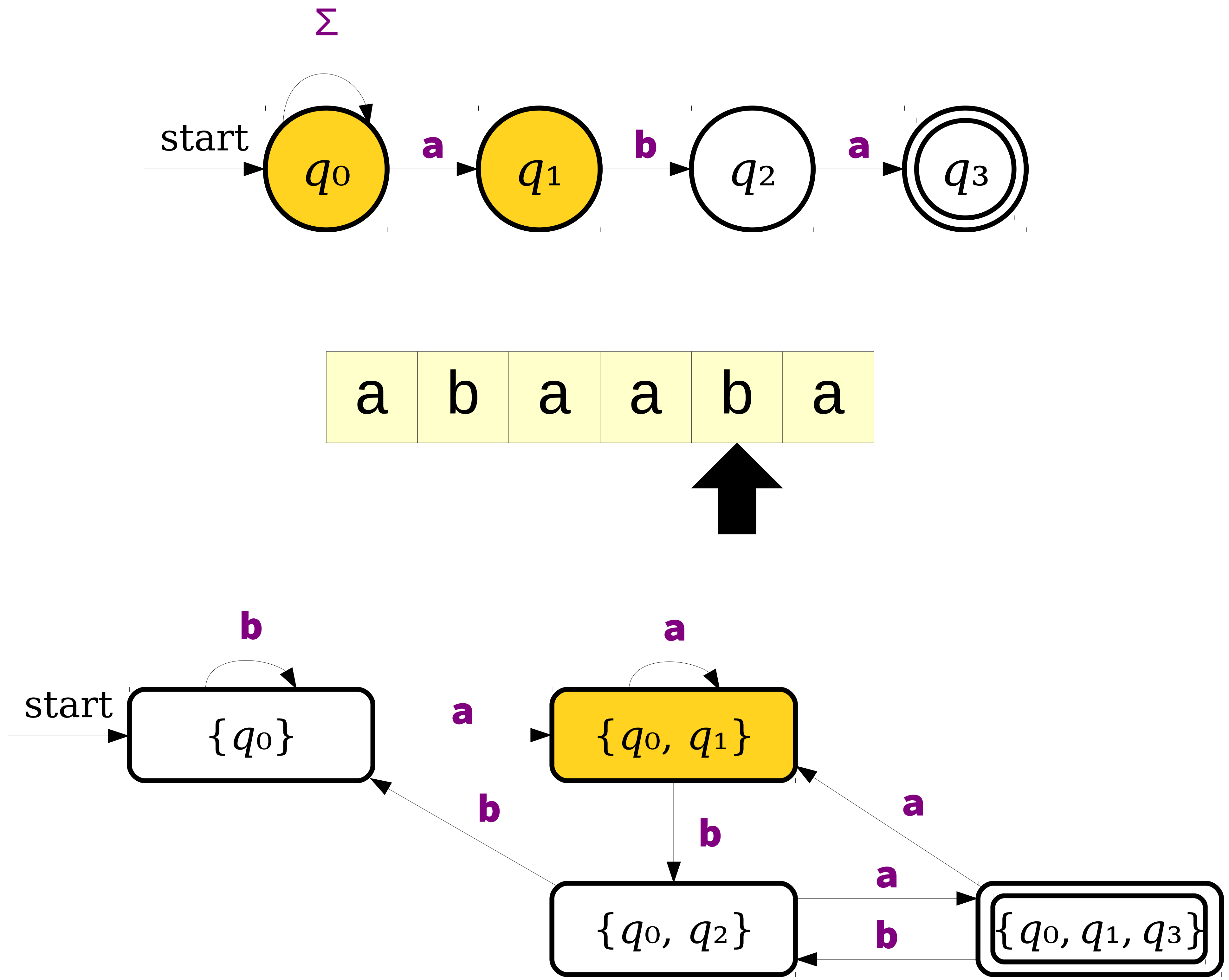

Σ
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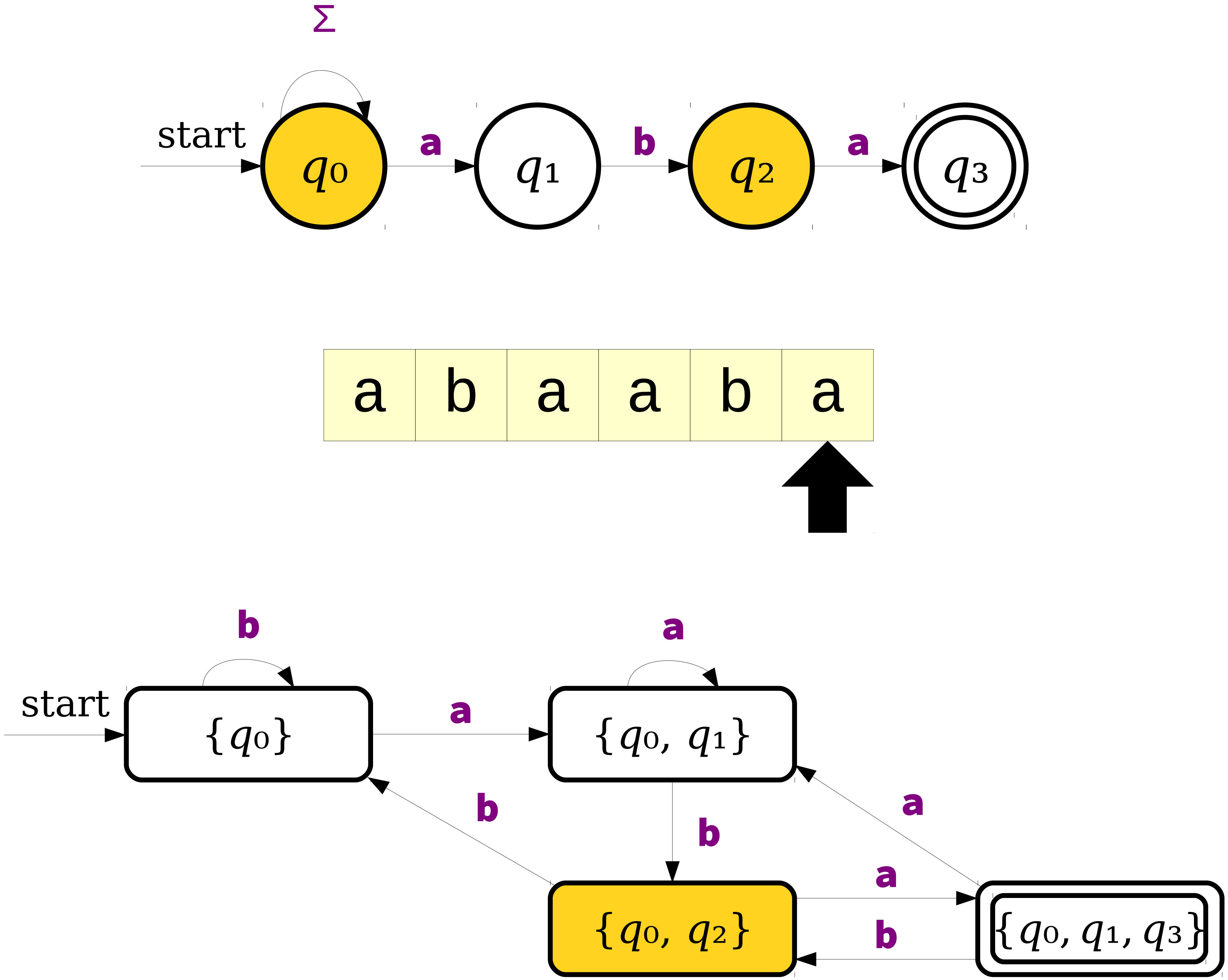

Σ
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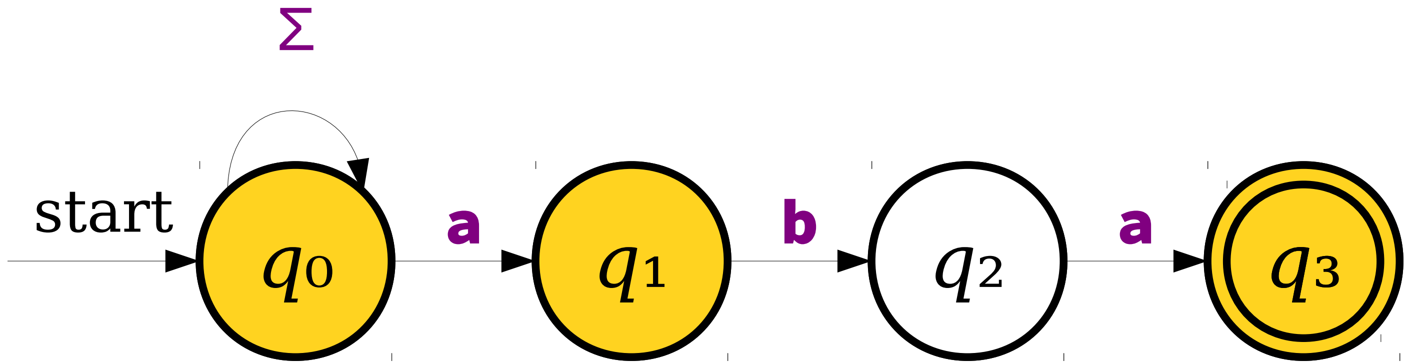
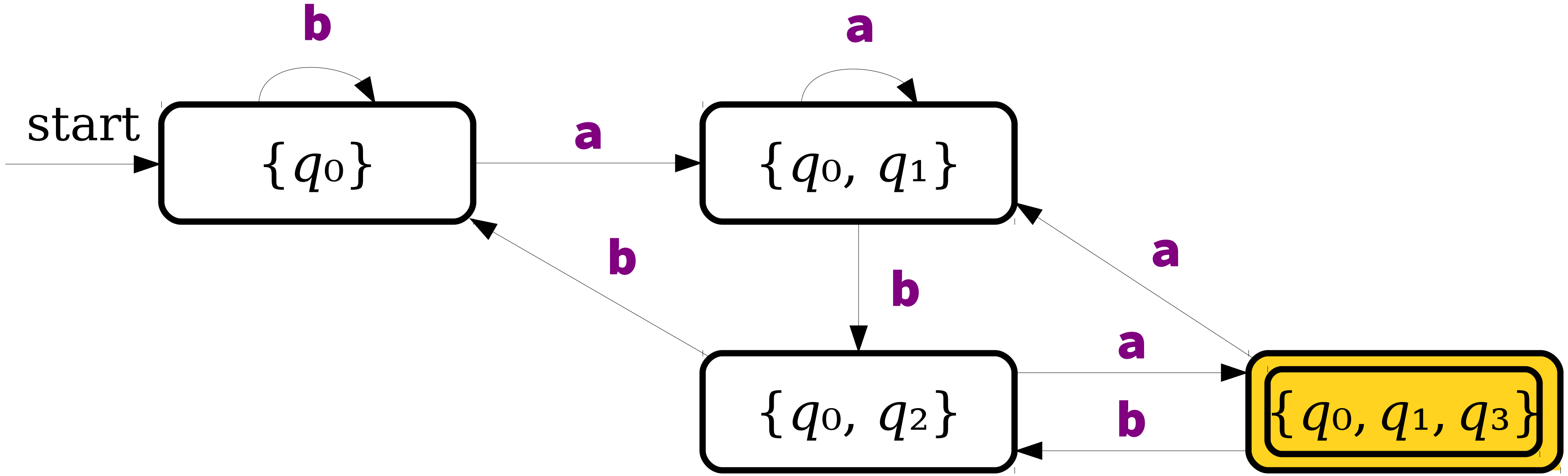


a	b	a	a	b	a



The Subset Construction

- This procedure for turning an NFA for a language L into a DFA for a language L is called the ***subset construction***.
  - It's sometimes called the ***powerset construction***; it's different names for the same thing!
- Intuitively:
  - Each state in the DFA corresponds to a set of states from the NFA.
  - Each transition in the DFA corresponds to what transitions would be taken in the NFA when using the massive parallel intuition.
  - The accepting states in the DFA correspond to which sets of states would be considered accepting in the NFA when using the massive parallel intuition.
- There's an online ***Guide to the Subset Construction*** with a more elaborate example involving ε-transitions and cases where the NFA dies; check that for more details.

The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
- ***Useful fact:*** $|\wp(S)| = 2^{|S|}$ for any finite set S.
- In the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- ***Question to ponder:*** Can you find a family of languages that have NFAs of size n, but no DFAs of size less than 2^n?

Why This Matters

- We now have two perspectives on regular languages:
  - Regular languages are languages accepted by DFAs.
  - Regular languages are languages accepted by NFAs.
- We can now reason about the regular languages in two different ways.

Properties of Regular Languages

The Union of Two Languages

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?

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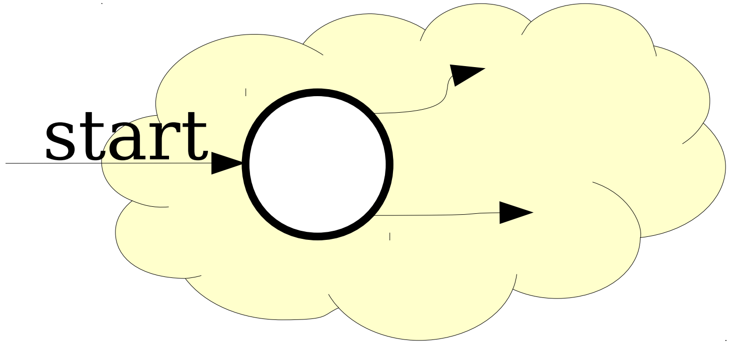
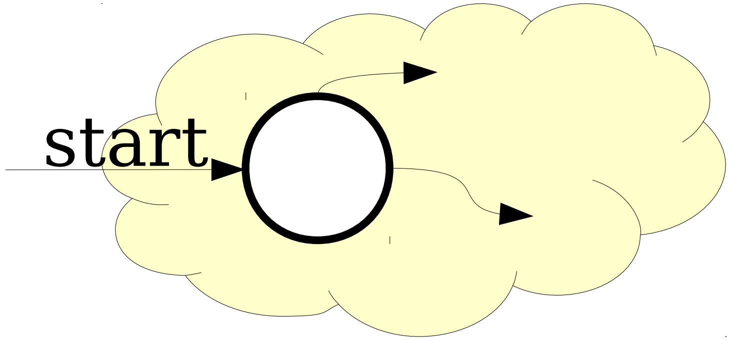


Machine for L_1



Machine for L_2

The Union of Two Languages

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
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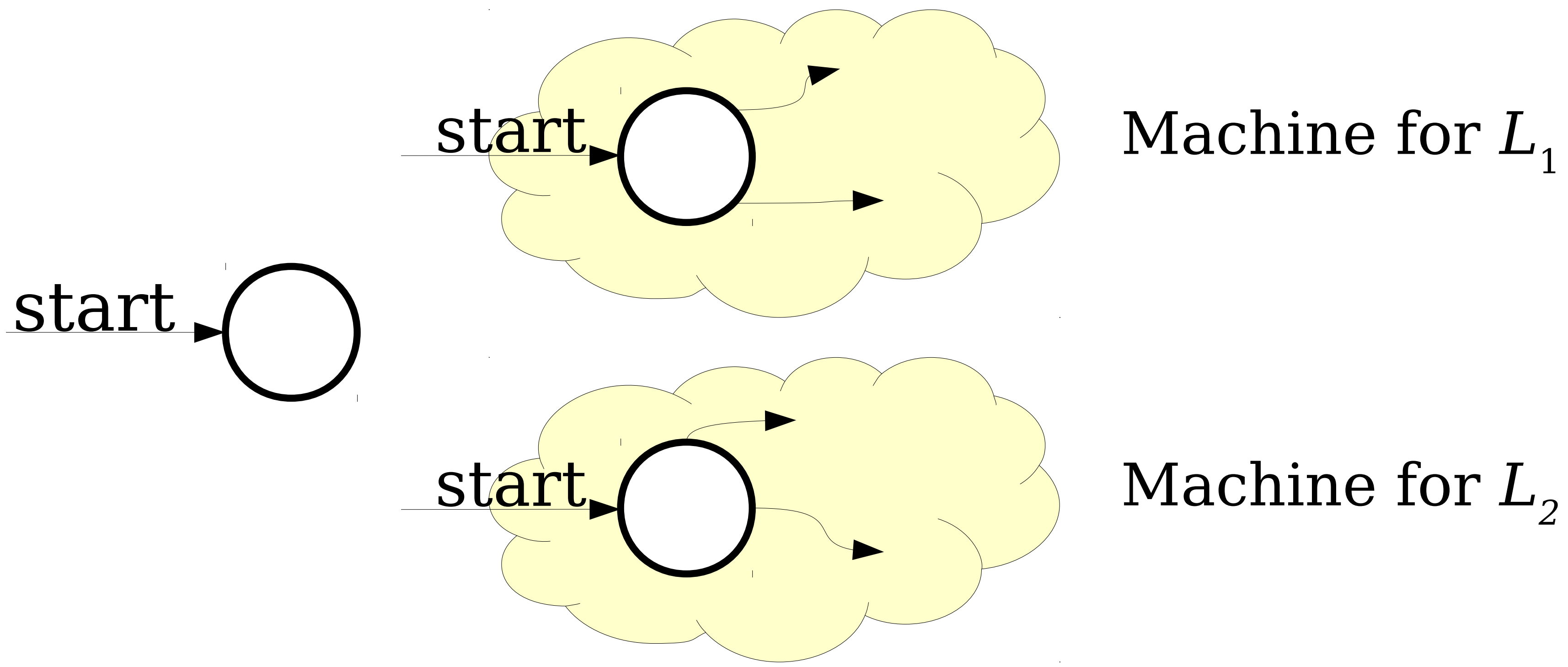

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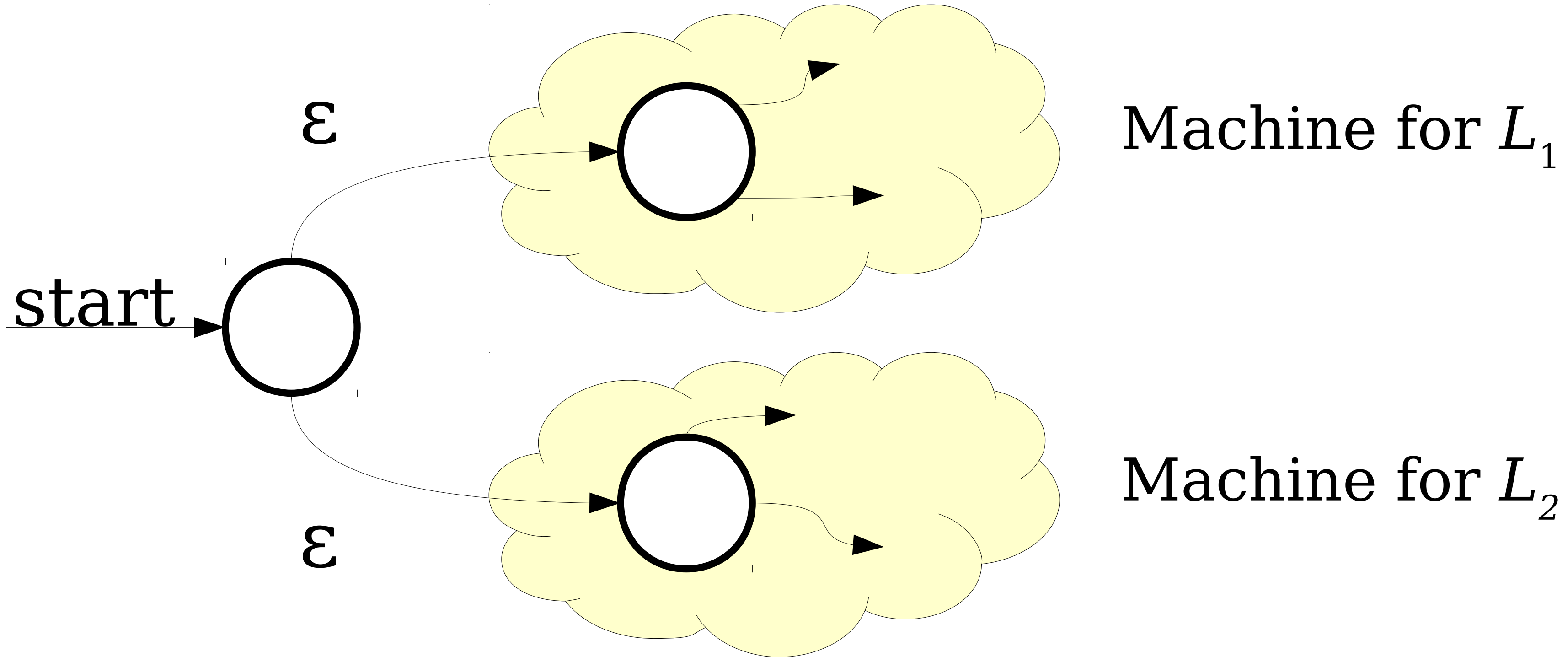

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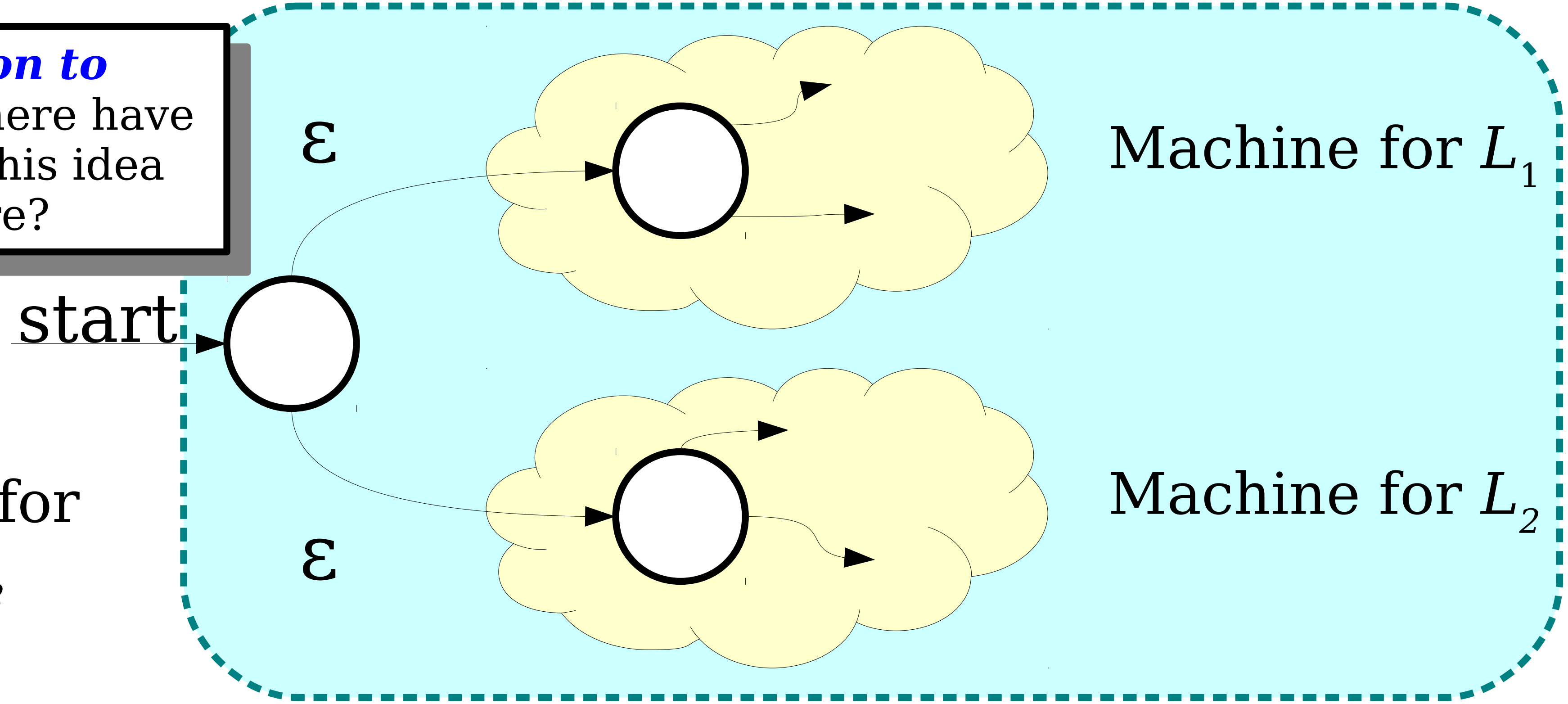
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?

Question to ponder: where have you seen this idea before?



Machine for $L_1 \cup L_2$

The Intersection of Two Languages

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?

The Intersection of Two Languages

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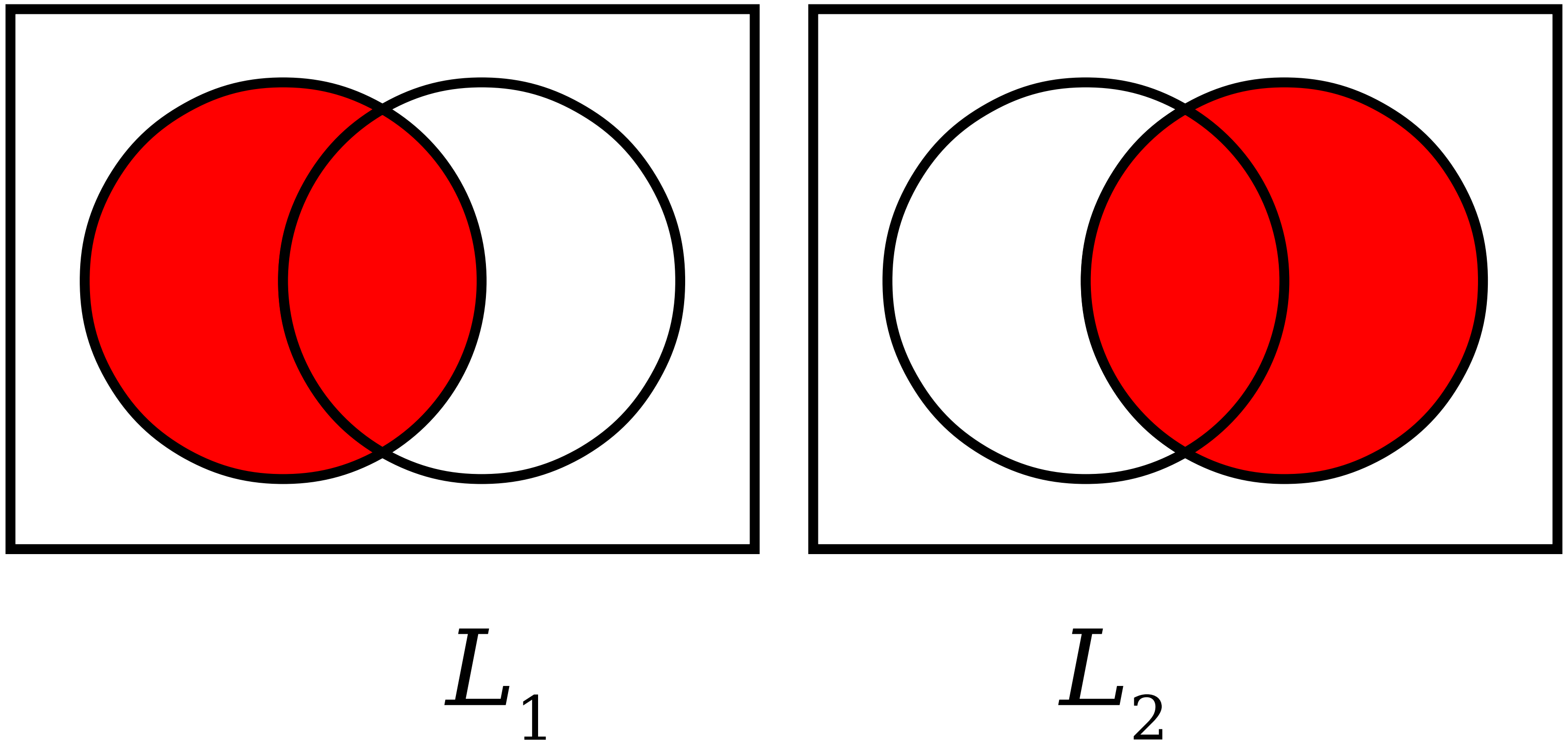

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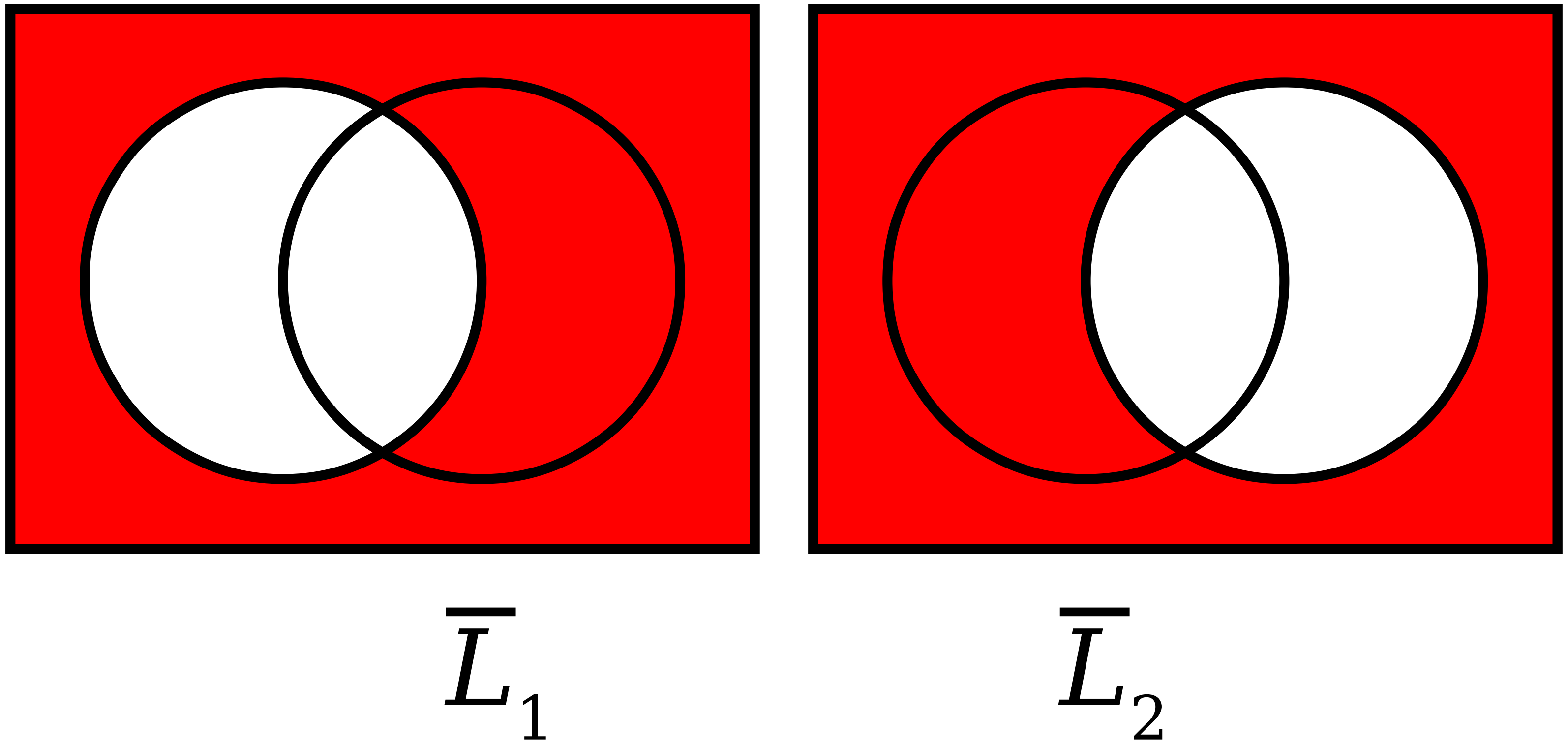

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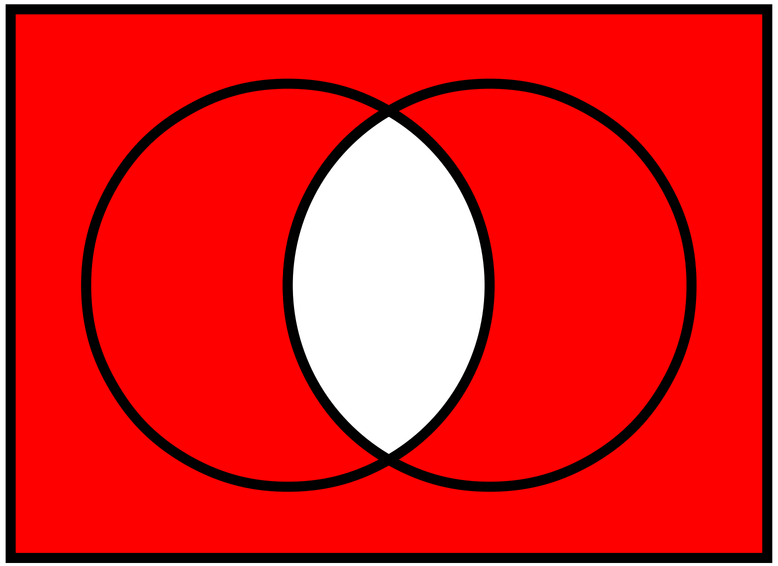

$$\overline{L}_1 \cup \overline{L}_2$$

The Intersection of Two Languages

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
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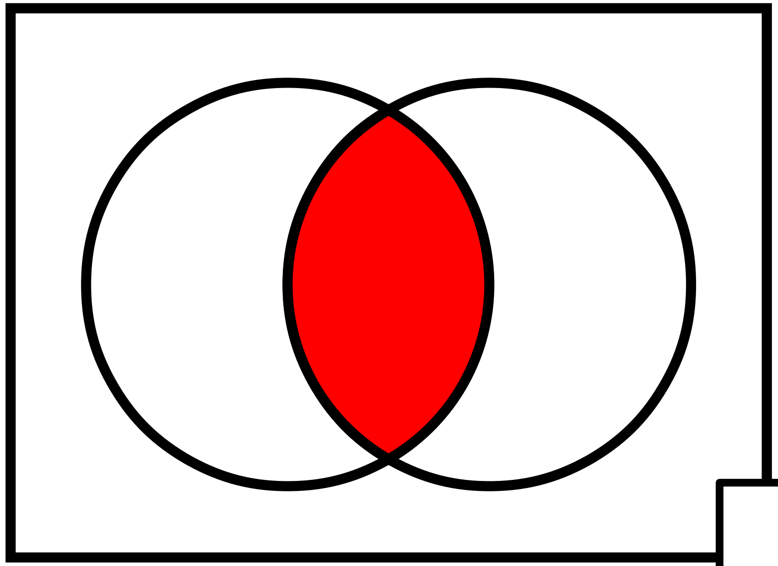

$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Hey, it's De Morgan's laws!

Concatenation

String Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, the ***concatenation*** of w and x, denoted $\boldsymbol{wx}$, is the string formed by tacking all the characters of x onto the end of w.
- Example: if w = **quo** and x = **kka**, the concatenation wx = **quokka**.
- This is analogous to the + operator for strings in many programming languages.
- Some facts about concatenation:
  - The empty string ε is the ***identity element*** for concatenation:

    $$w\varepsilon = \varepsilon w = w$$

  - Concatenation is ***associative***:

    $$wxy = w(xy) = (wx)y$$

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \land x \in L_2 \}$$

Concatenation Example

- Let Σ = { **a**, **b**, …, **z**, **A**, **B**, …, **Z** } and consider these languages over Σ:
  - ***Noun*** = { **Puppy**, **Rainbow**, **Whale**, … }
  - ***Verb*** = { **Hugs**, **Juggles**, **Loves**, … }
  - ***The*** = { **The** }
- The language ***TheNounVerbTheNoun*** is
  - { **ThePuppyHugsTheWhale**, **TheWhaleLovesTheRainbow**, **TheRainbowJugglesTheRainbow**, … }

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \land x \in L_2 \}$$

- Two views of L_1L_2:
  - The set of all strings that can be made by concatenating a string in L_1 with a string in L_2.
  - The set of strings that can be split into two pieces: a piece from L_1 and a piece from L_2.

This is closely related to, but different than, the Cartesian product.

Question to ponder: In what ways are concatenations similar to Cartesian products? In what ways are they different?

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?
- ***Idea:***

Concatenating Regular Languages

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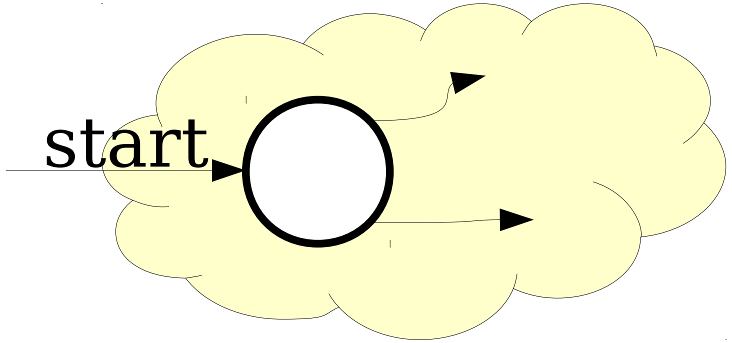
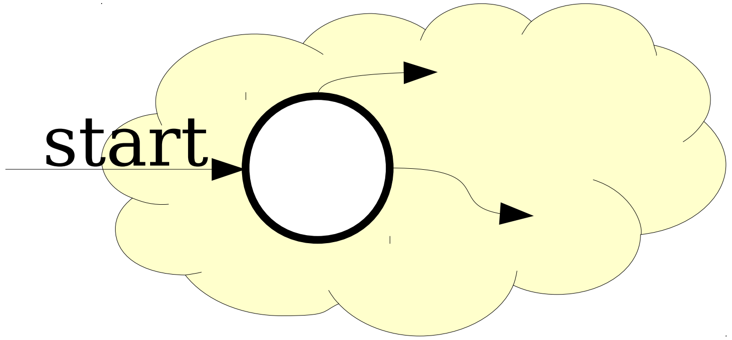


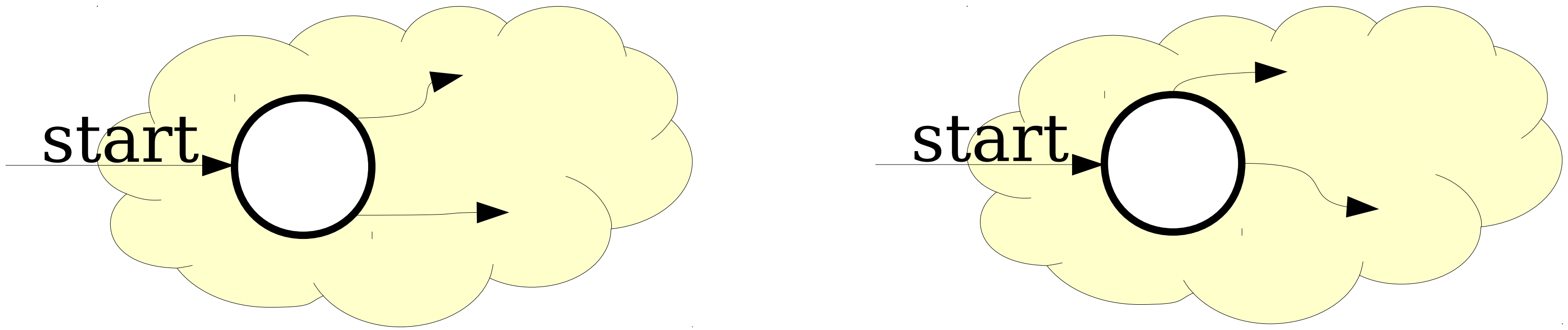
Machine for L_1



Machine for L_2

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?
- ***Idea:***



Machine for L_1 Machine for L_2

b	**o**	**o**	**k**	**k**	**e**	**e**	**p**	**e**	**r**

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
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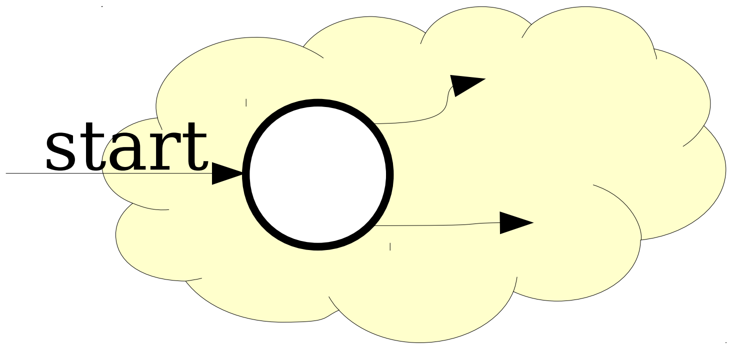
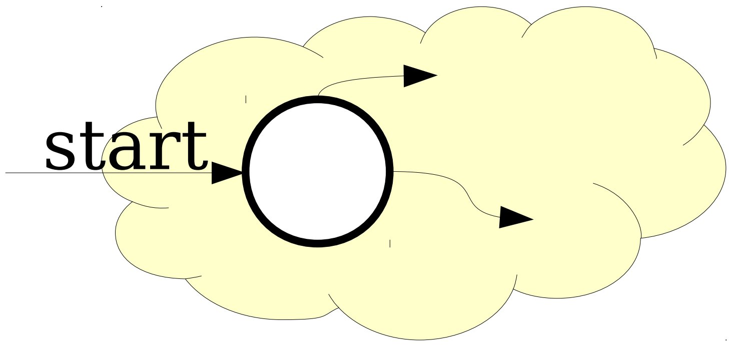


Machine for L_1



Machine for L_2

b	**o**	**o**	**k**	**k**	**e**	**e**	**p**	**e**	**r**

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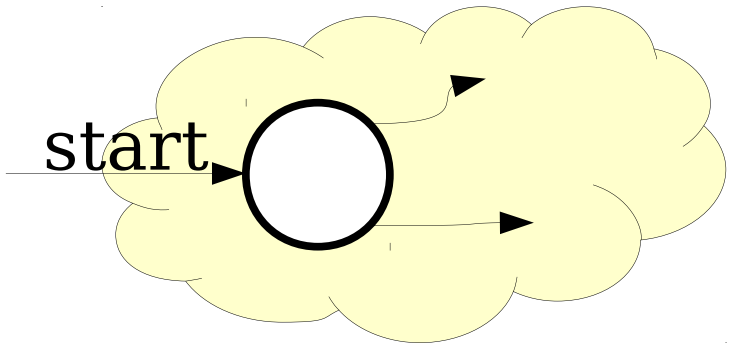
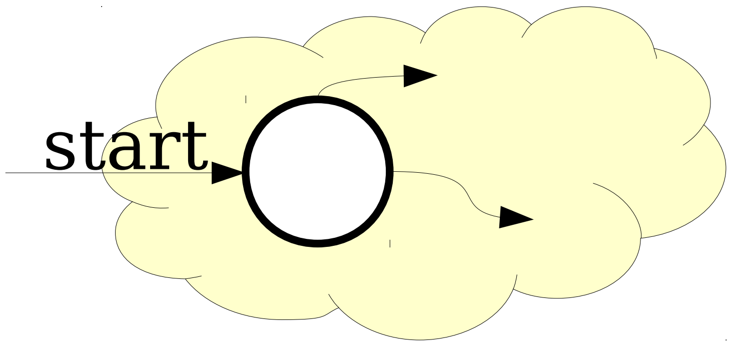


Machine for L_1



Machine for L_2

b	o	o	k

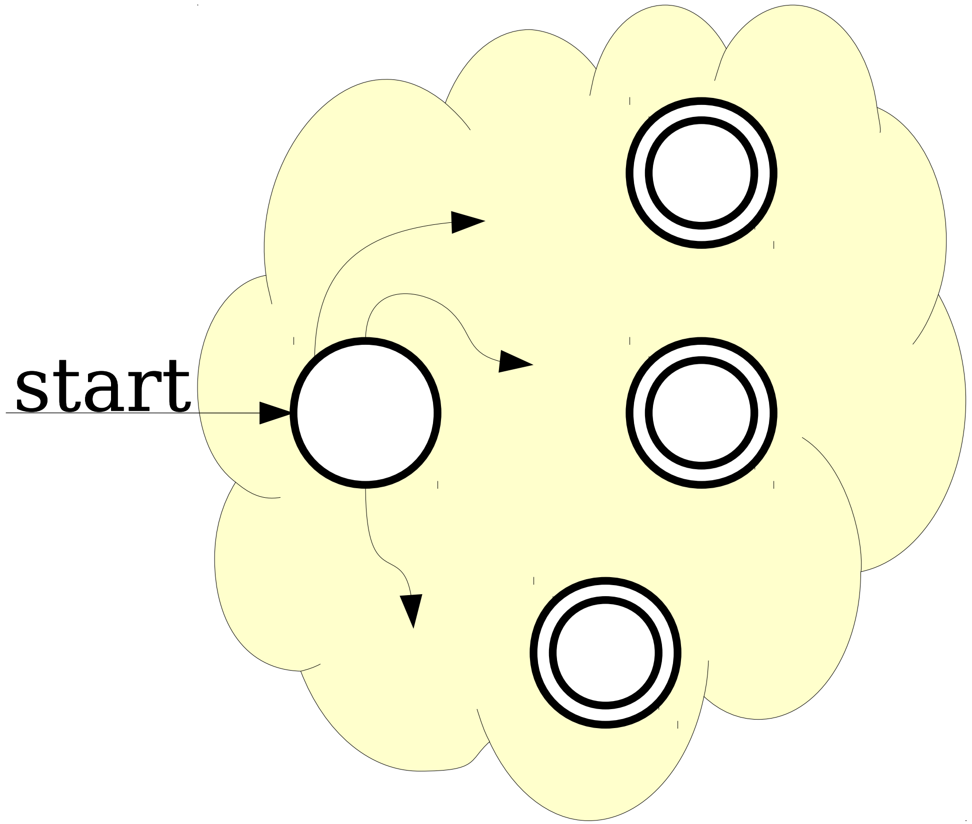
k	e	e	p	e	r

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?
- ***Idea***:
  - Run a DFA/NFA for L_1 on w.
  - Whenever it reaches an accepting state, optionally hand the rest of w to a DFA/NFA for L_2.
  - If the automaton for L_2 accepts the rest, $w \in L_1L_2$.
  - If the automaton for L_2 rejects the remainder, the split was incorrect.

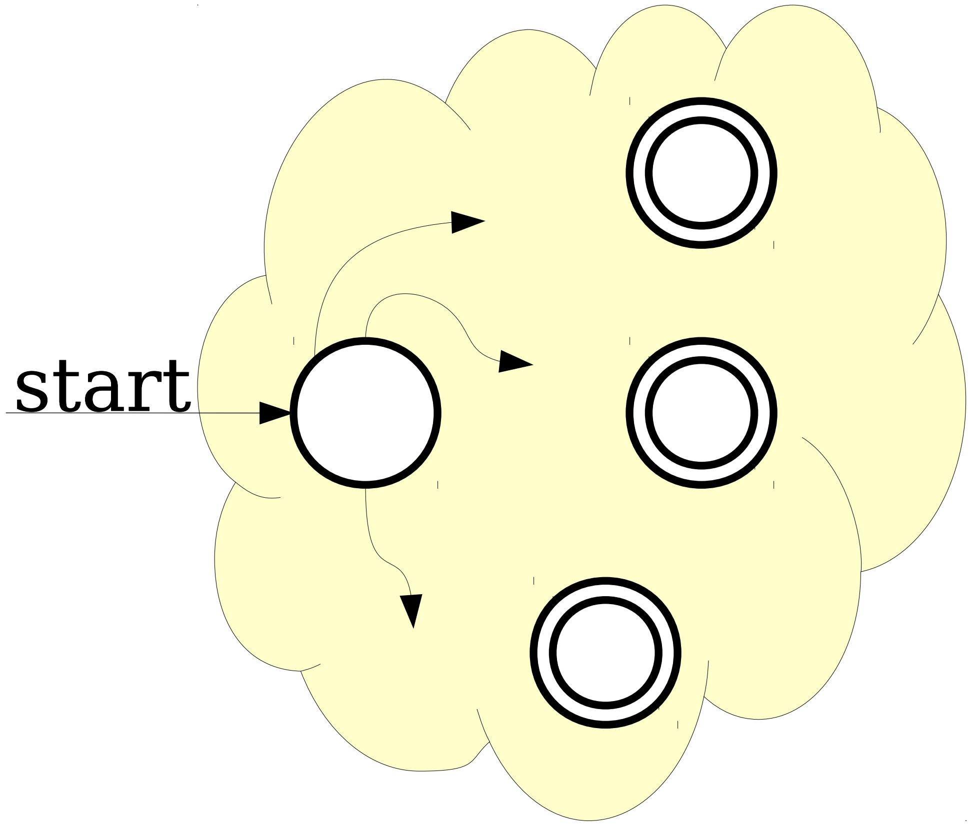
Concatenating Regular Languages

Concatenating Regular Languages

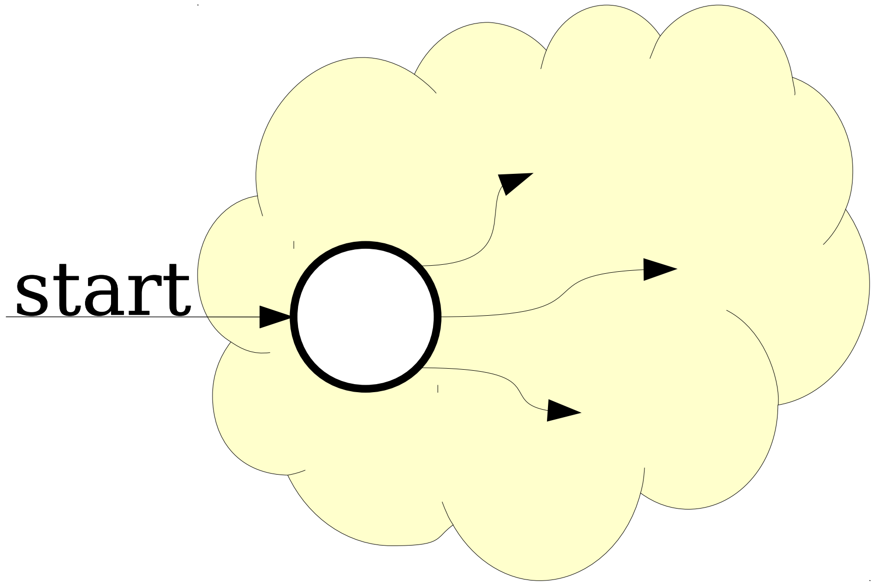


Machine for L_1

Concatenating Regular Languages

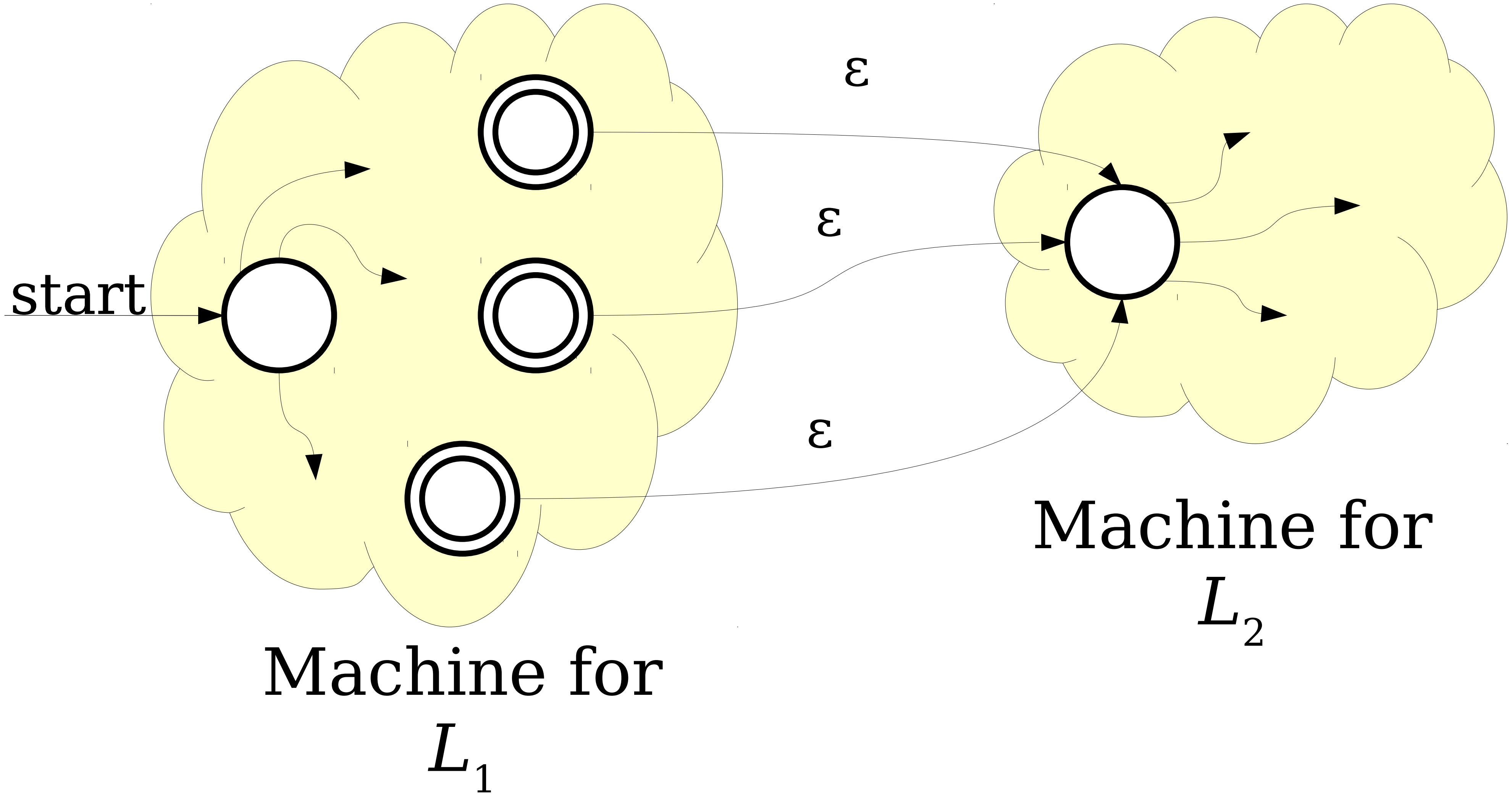


Machine for
L_1

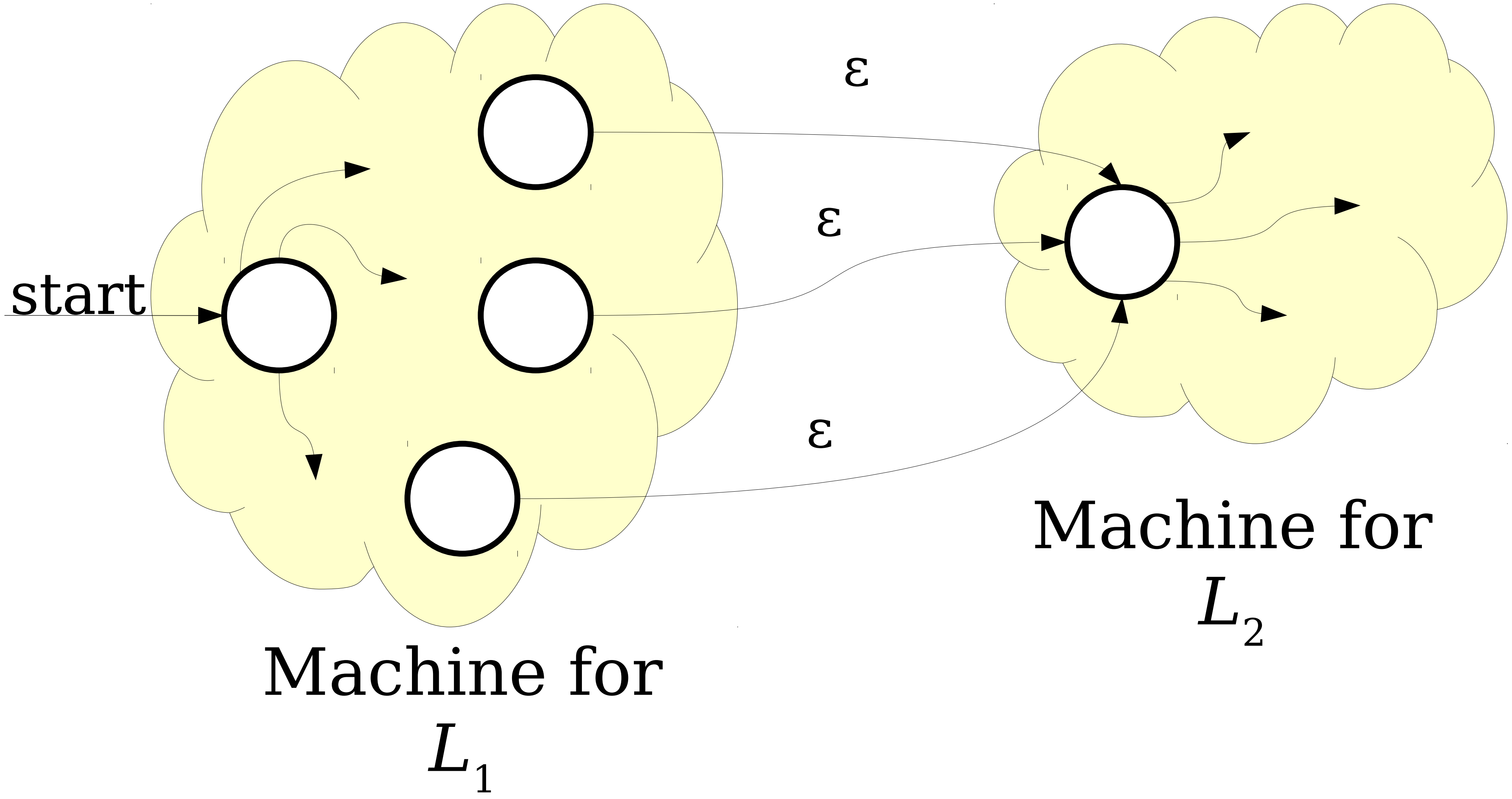


Machine for
L_2

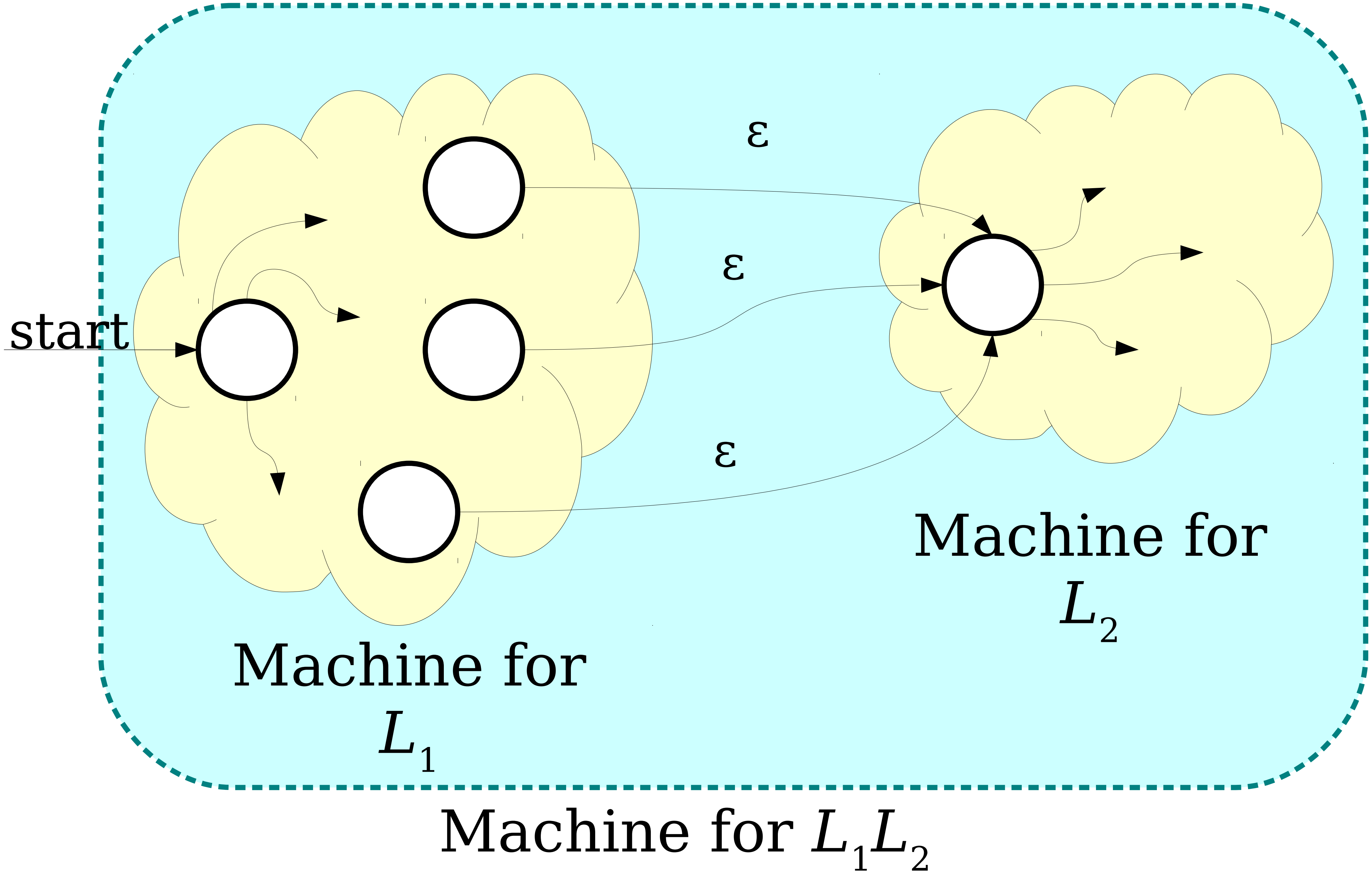
Concatenating Regular Languages



Concatenating Regular Languages



Concatenating Regular Languages



Lots and Lots of Concatenation

- Consider the language L = { **aa**, **b** }
- LL is the set of strings formed by concatenating pairs of strings in L.

{ **aaaa**, **aab**, **baa**, **bb** }

- LLL is the set of strings formed by concatenating triples of strings in L.

{ **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

{ **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**, **aabaab**, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**, **baabb**, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

Language Exponentiation

- We can define what it means to “exponentiate” a language as follows:
- $L^0 = \{\varepsilon\}$
  - Intuition: The only string you can form by gluing no strings together is the empty string.
  - Notice that $\{\varepsilon\} \neq \emptyset$. Can you explain why?
- $L^{n+1} = LL^n$
  - Idea: Concatenating (n+1) strings together works by concatenating n strings, then concatenating one more.
- ***Question to ponder:*** Why define $L^0 = \{\varepsilon\}$?
- ***Question to ponder:*** What is $\emptyset^0$?

The Kleene Star

The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}$$

- Mathematically:

$$w \in L^* \quad \leftrightarrow \quad \exists n \in \mathbb{N}.\ w \in L^n$$

- Intuitively, L^* is the language all possible ways of concatenating zero or more strings in L together, possibly with repetition.
- ***Question to ponder:*** What is Ø*?

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

…

}

Think of L^* as the set of strings you can make if you have a collection of stamps – one for each string in *L* – and you form every possible string that can be made from those stamps.

Reasoning about Infinity

- If L is regular, is L^* necessarily regular?
- **⚠ A Bad Line of Reasoning: ⚠**
  - $L^0 = \{ \varepsilon \}$ is regular.
  - $L^1 = L$ is regular.
  - $L^2 = LL$ is regular
  - $L^3 = L(LL)$ is regular
  - …
  - Regular languages are closed under union.
  - So the union of all these languages is regular.

Reasoning about Infinity

0 is finite

Reasoning about Infinity

1 is finite

Reasoning about Infinity

2 is finite

Reasoning about Infinity

3 is finite

Reasoning about Infinity

4 is finite

Reasoning about Infinity

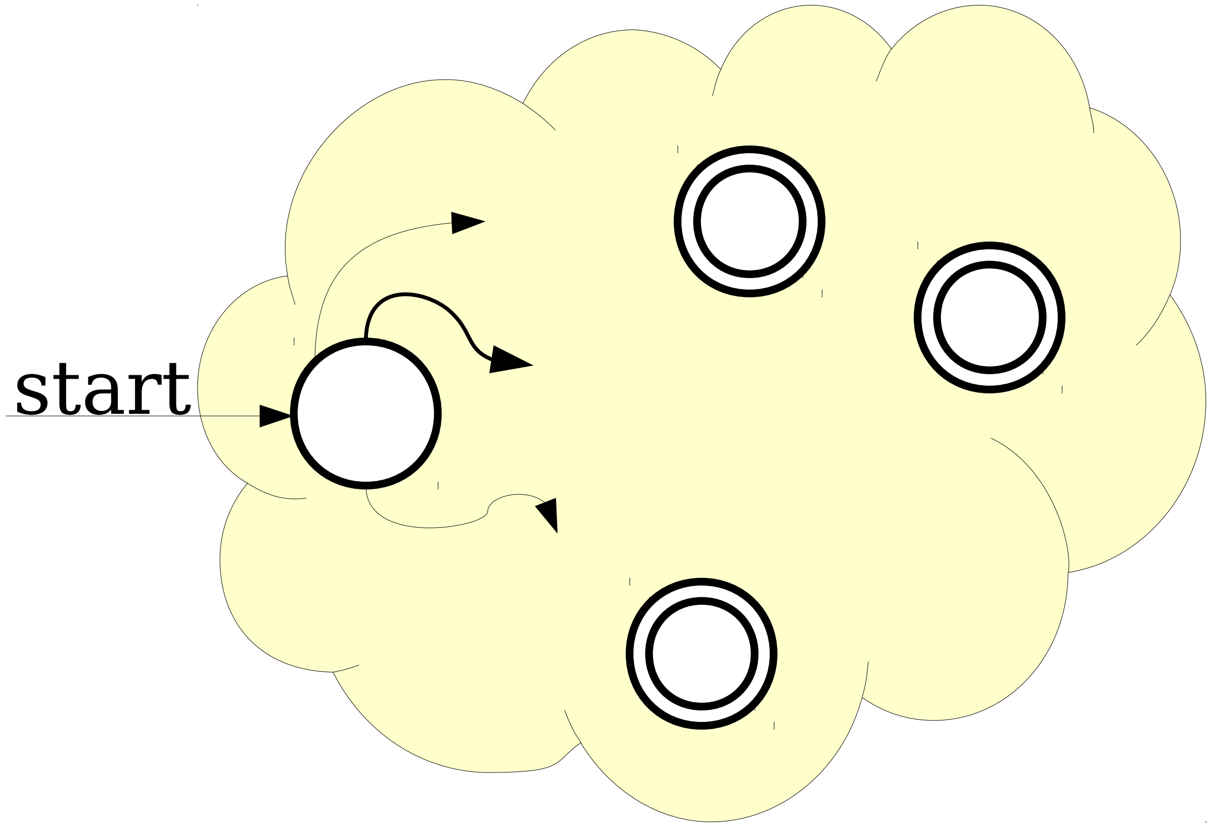
∞ is finite
^ not

Reasoning About the Infinite

- If a series of finite objects all have some property, the “limit” of that process *does not* necessarily have that property.
- In general, it is not safe to conclude that some property that always holds in the finite case must hold in the infinite case.
  - (This is why calculus is interesting).
- So our earlier argument ($L^* = L^0 \cup L^1 \cup \ldots$) isn’t going to work.
- We need a different line of reasoning.

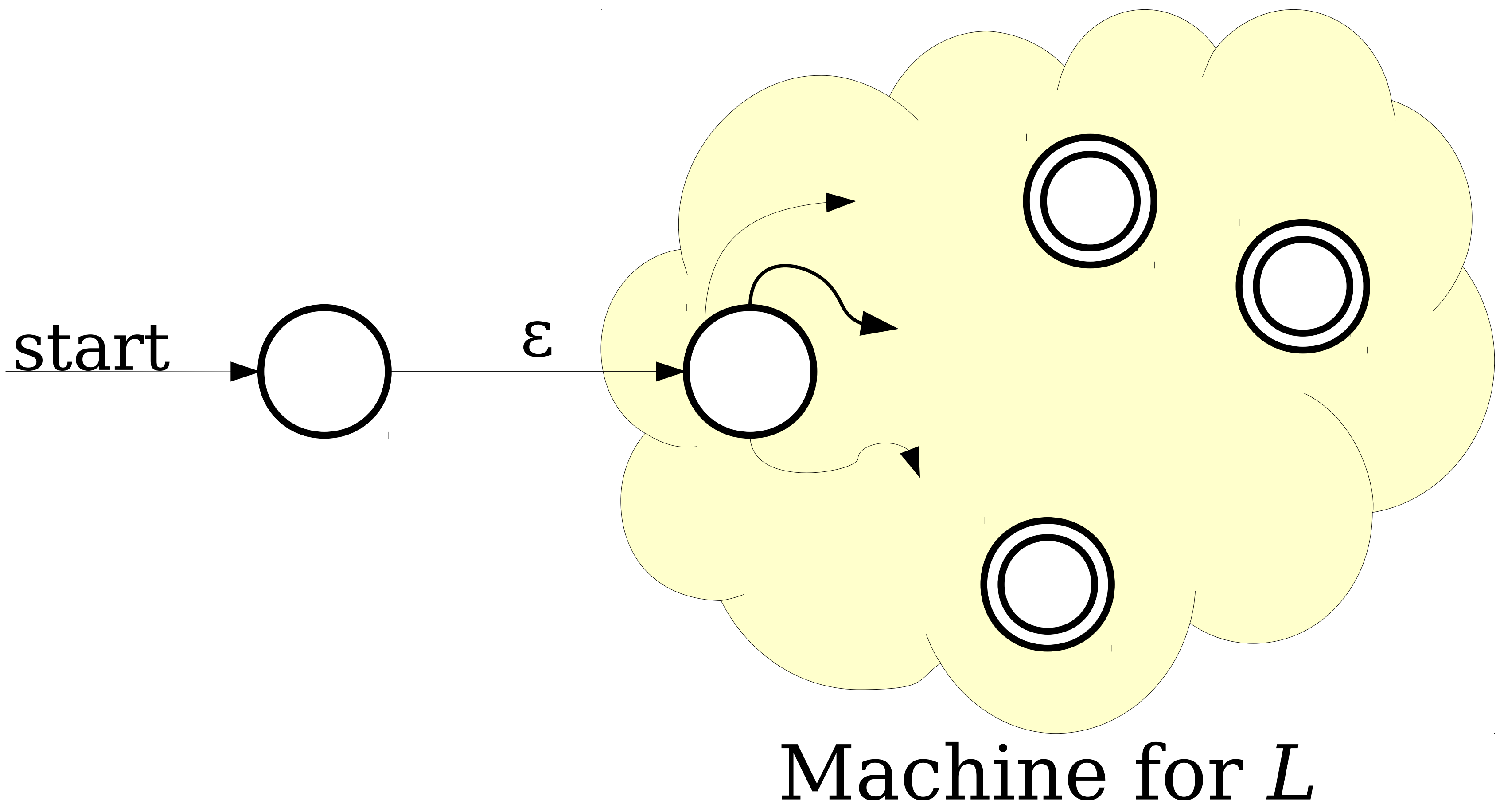
Idea: Can we directly convert an NFA for language L to an NFA for language L^*?

The Kleene Star

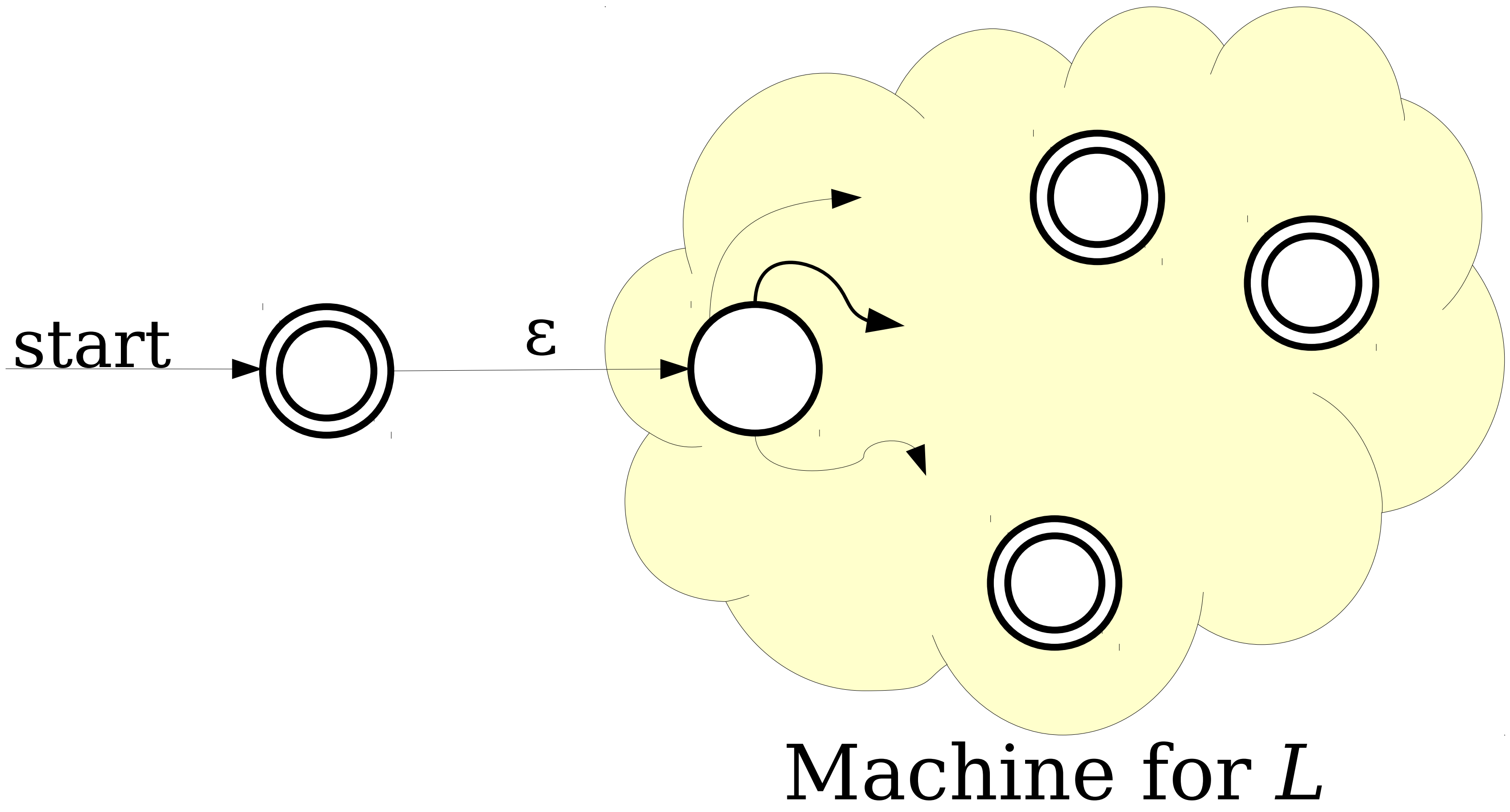


Machine for L

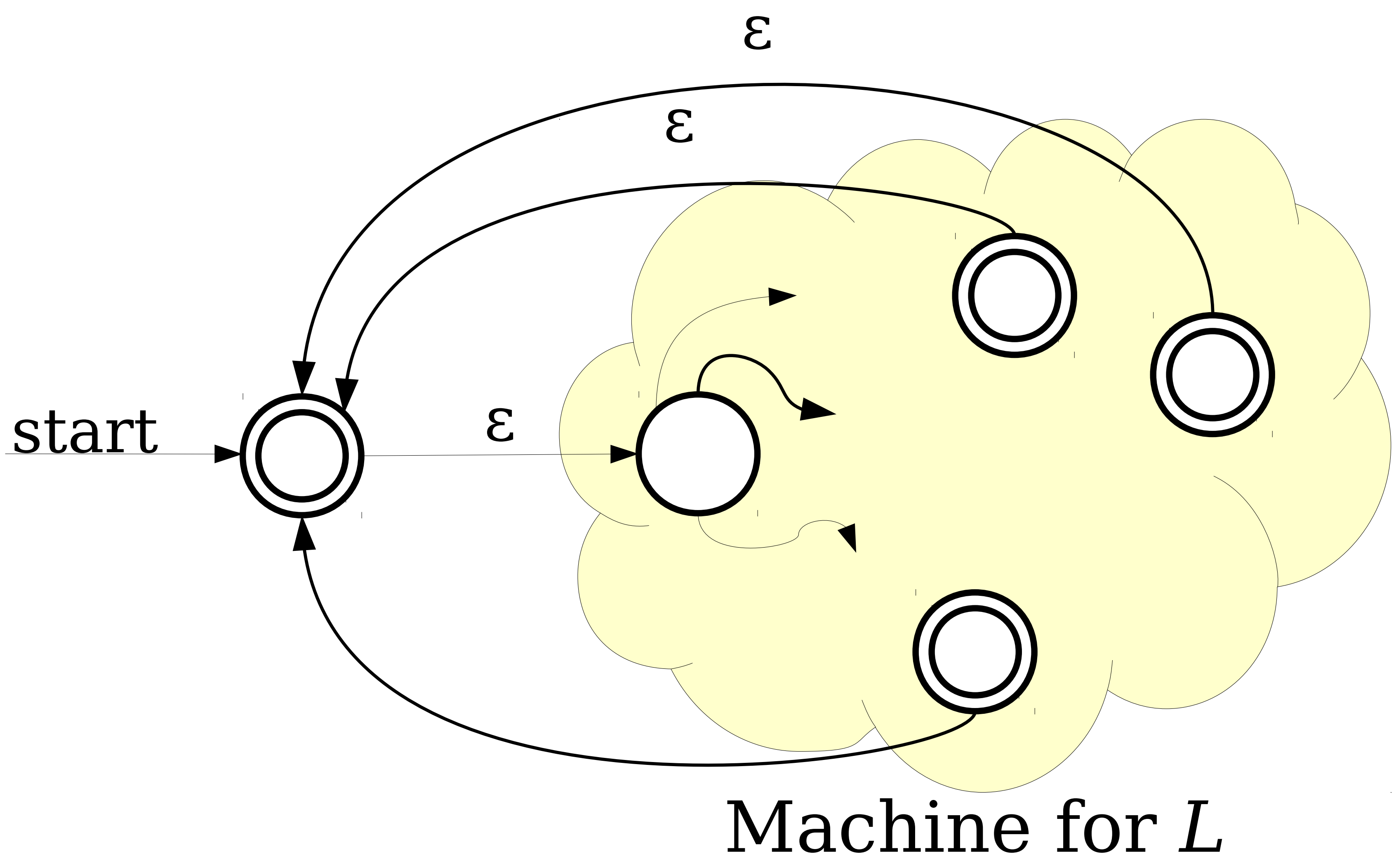
The Kleene Star



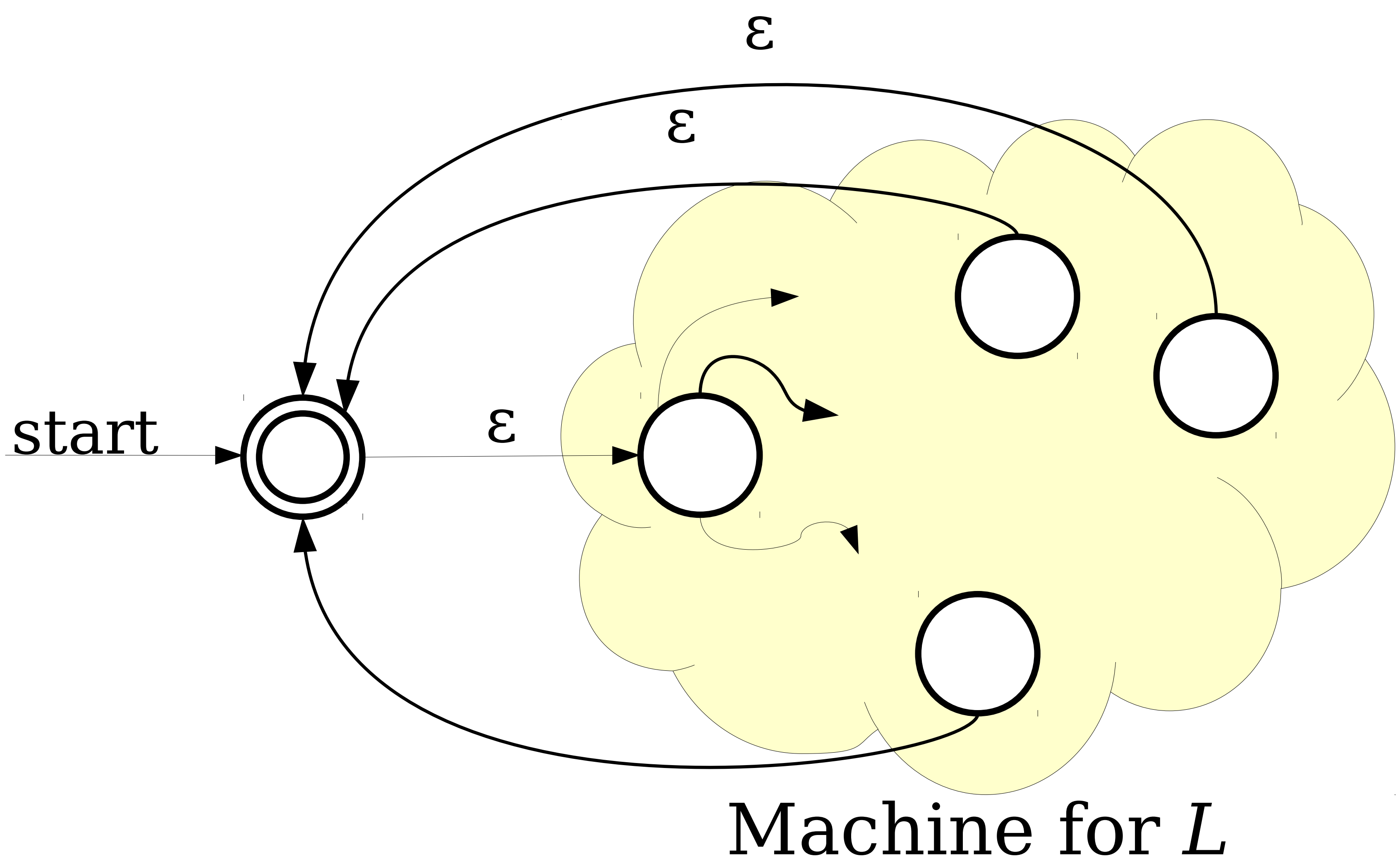
The Kleene Star



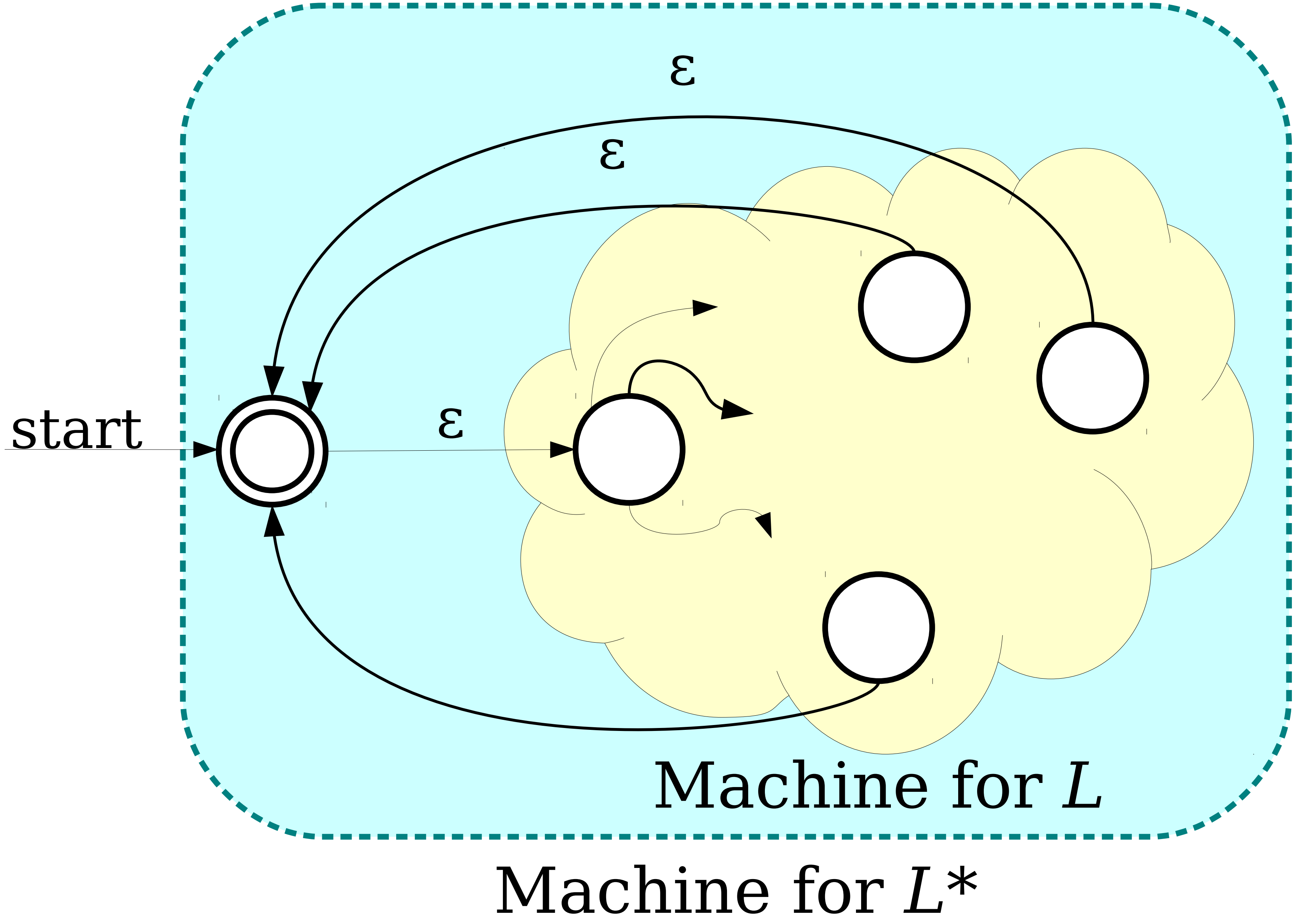
The Kleene Star



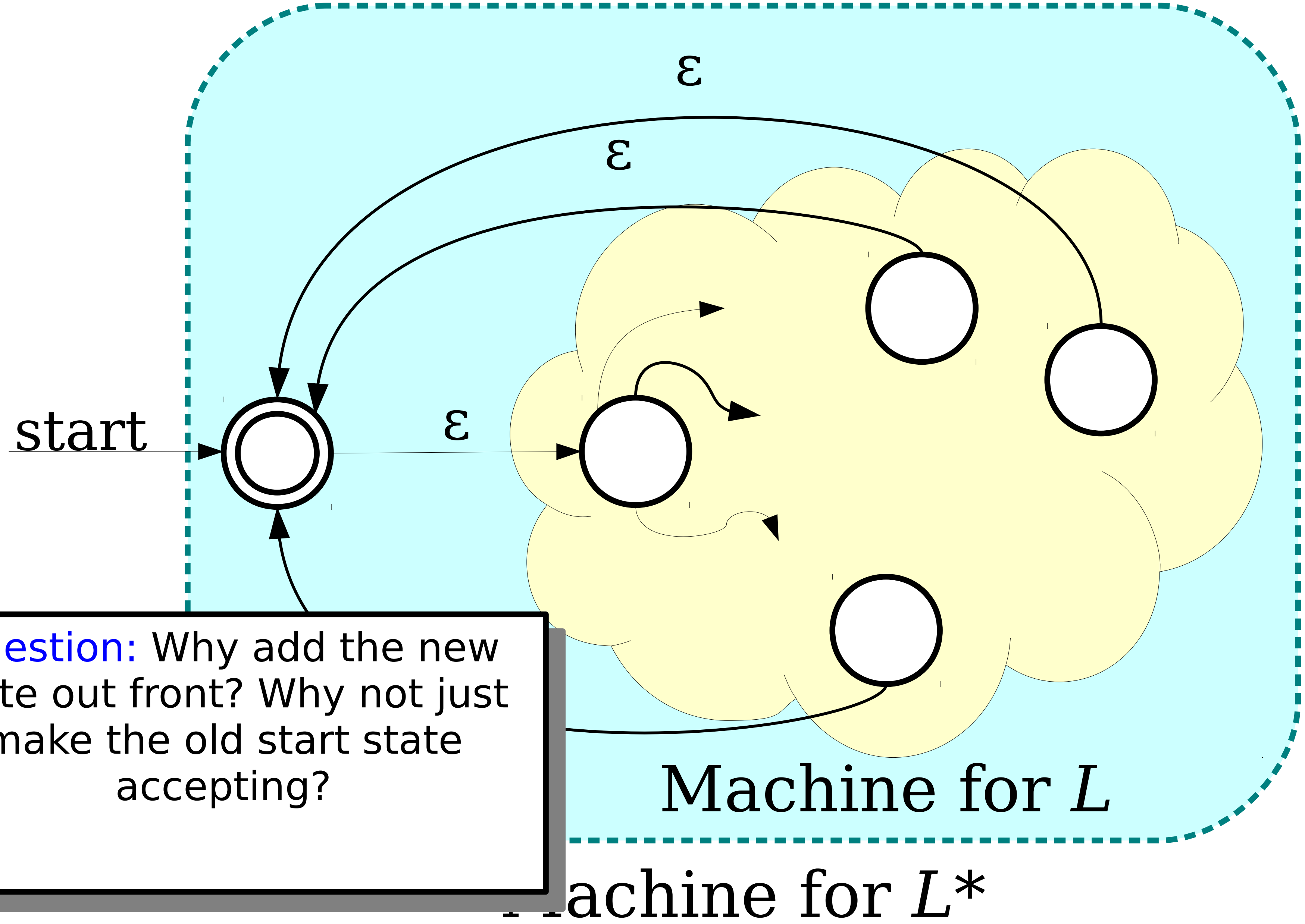
The Kleene Star



The Kleene Star



The Kleene Star



Question: Why add the new state out front? Why not just make the old start state accepting?

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - L_1L_2
  - L_1^*
- These properties are called ***closure properties of the regular languages***.

Next Time

- ***Regular Expressions***
  - Building languages from the ground up!
- ***Thompson's Algorithm***
  - A UNIX Programmer in Theoryland.
- ***Kleene's Theorem***
  - From machines to programs!